



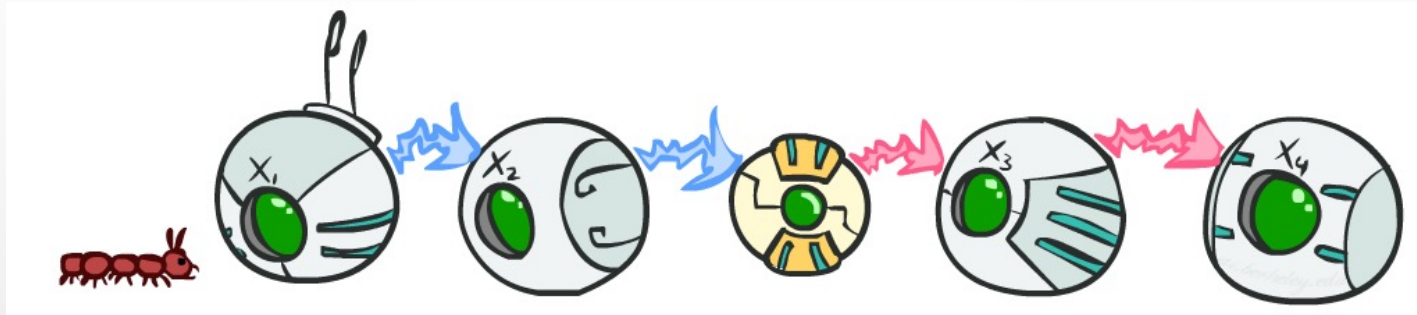
# Artificial Intelligence CE-417, Group 1 Computer Eng. Department Sharif University of Technology

Fall 2023

By Mohammad Hossein Rohban, Ph.D.

Courtesy: Most slides are adopted from CSE-573 (Washington U.), original slides for the textbook, and CS-188 (UC. Berkeley).

# Temporal Probability Models



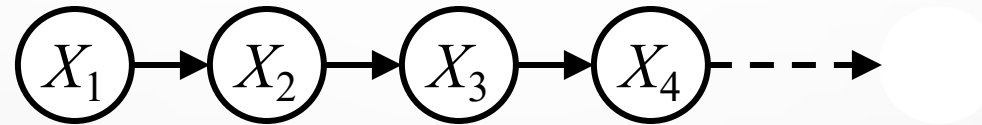
# Markov Models

# Reasoning over Time or Space

- Often, we want to **reason about a sequence** of observations
  - Speech recognition
  - Robot localization
  - User attention
  - Medical monitoring
- Need to introduce time (or space) into our models

# Markov Models

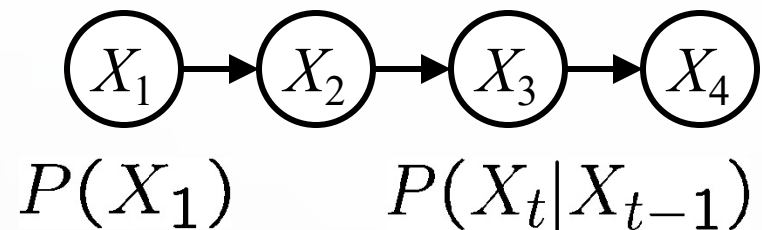
- Value of  $X$  at a given time is called the **state**



$$P(X_1) \quad P(X_t|X_{t-1})$$

- Parameters: called **transition probabilities** or dynamics, specify how the state evolves over time (also, initial state probabilities)
- Stationarity assumption: transition probabilities the same at all times
- Same as MDP transition model, but no choice of action
- A "**growable**" BN (can always use BN methods if we truncate to fixed length)

# Joint Distribution of a Markov Model



- Joint distribution:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$

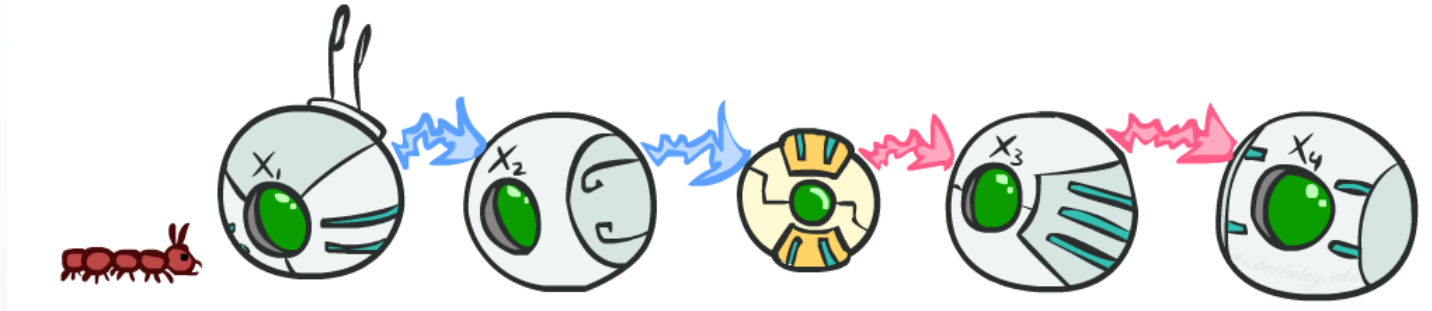
- More generally:

$$\begin{aligned} P(X_1, X_2, \dots, X_T) &= P(X_1)P(X_2|X_1)P(X_3|X_2) \dots P(X_T|X_{T-1}) \\ &= P(X_1) \prod_{t=2}^T P(X_t|X_{t-1}) \end{aligned}$$

- Questions to be resolved:

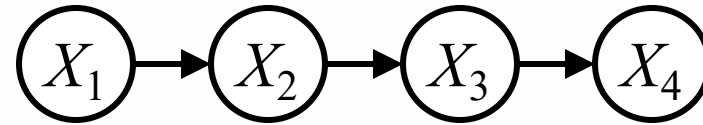
- Does this indeed define a joint distribution?
- Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

# Markov Assumption: Conditional Independence



- **Basic conditional independence:**
  - Past and future independent given the present
  - Each time step only depends on the previous
  - This is called the (first order) Markov property

# Chain Rule and Markov Models



- From the chain rule, every joint distribution over  $X_1, X_2, X_3, X_4$  can be written as:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)P(X_4|X_1, X_2, X_3)$$

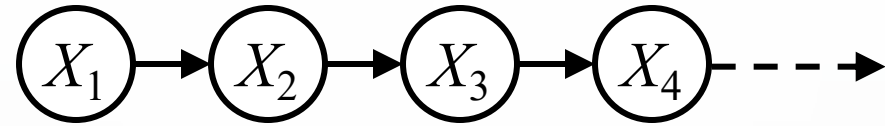
- Assuming that  $X_3 \perp\!\!\!\perp X_1 \mid X_2$  and  $X_4 \perp\!\!\!\perp X_1, X_2 \mid X_3$

Results in the expression posited on the previous slide:

$$P(X_1, X_2, X_3, X_4) = P(X_1)P(X_2|X_1)P(X_3|X_2)P(X_4|X_3)$$



# Chain Rule and Markov Models: General Case



- From the chain rule, every joint distribution over  $X_1, X_2, \dots, X_T$  can be written as:

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_1, X_2, \dots, X_{t-1})$$

- Assuming that for all  $t$ :

$$X_t \perp\!\!\!\perp X_1, \dots, X_{t-2} \mid X_{t-1}$$

Gives us the expression posited on the earlier slide:

$$P(X_1, X_2, \dots, X_T) = P(X_1) \prod_{t=2}^T P(X_t | X_{t-1})$$

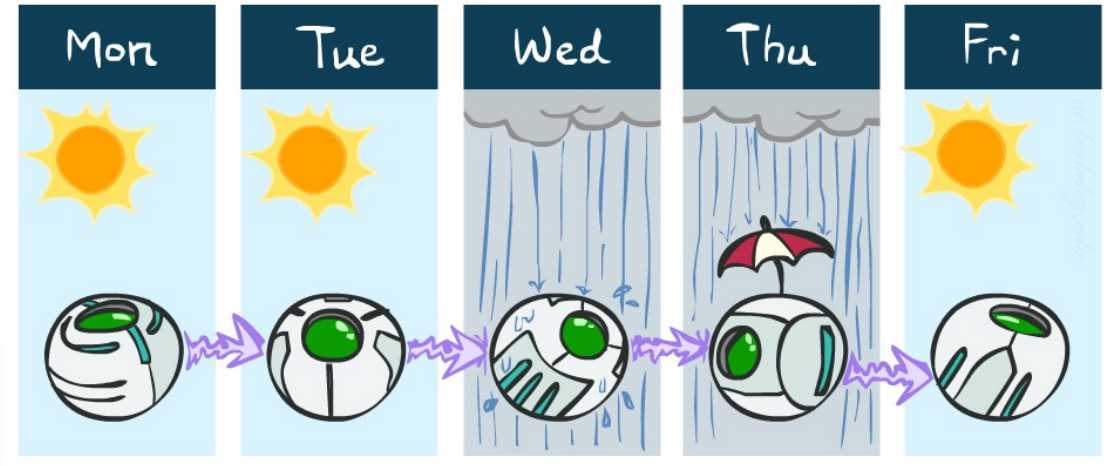
# Example Markov Chain: Weather

- States:  $X = \{\text{rain, sun}\}$

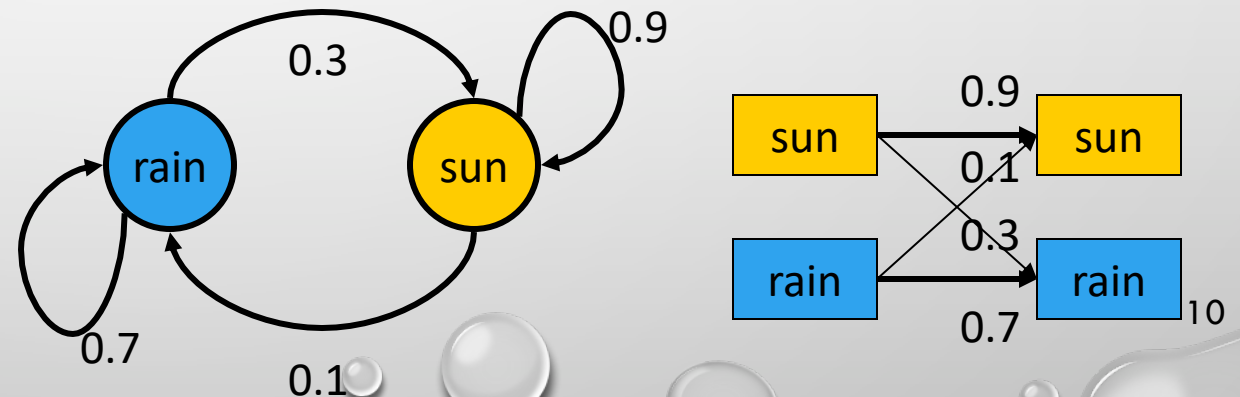
- Initial distribution: 1.0 sun

- CPT  $P(X_t | X_{t-1})$ :

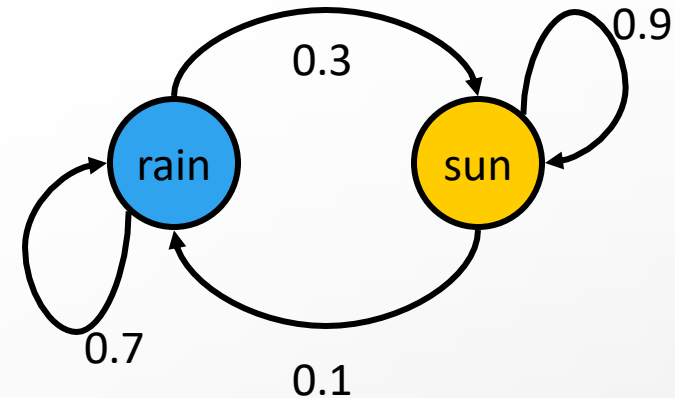
$X_{t-1}$	$X_t$	$P(X_t   X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7



Two new ways of representing the same CPT



# Example Markov Chain: Weather



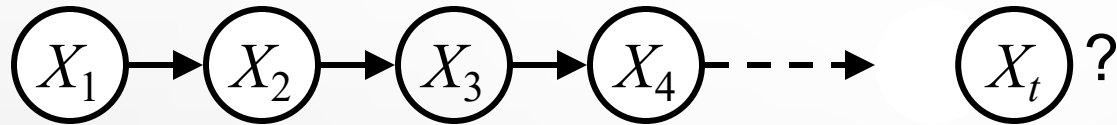
- Initial distribution: 1.0 sun
  - We know:  $P(X_1), P(X_t|X_{t-1})$

- What is the probability distribution after one step?

$$\begin{aligned} P(X_2 = sun) &= \sum_{x_1} P(x_1, X_2 = sun) = \sum_{x_1} P(X_2 = sun | x_1) P(x_1) = \\ &P(X_2 = sun | X_1 = sun) P(X_1 = sun) + P(X_2 = sun | X_1 = rain) P(X_1 = rain) = \\ &0.9 \times 1.0 + 0.3 \times 0.0 = 0.9 \end{aligned}$$

# Mini-Forward Algorithm

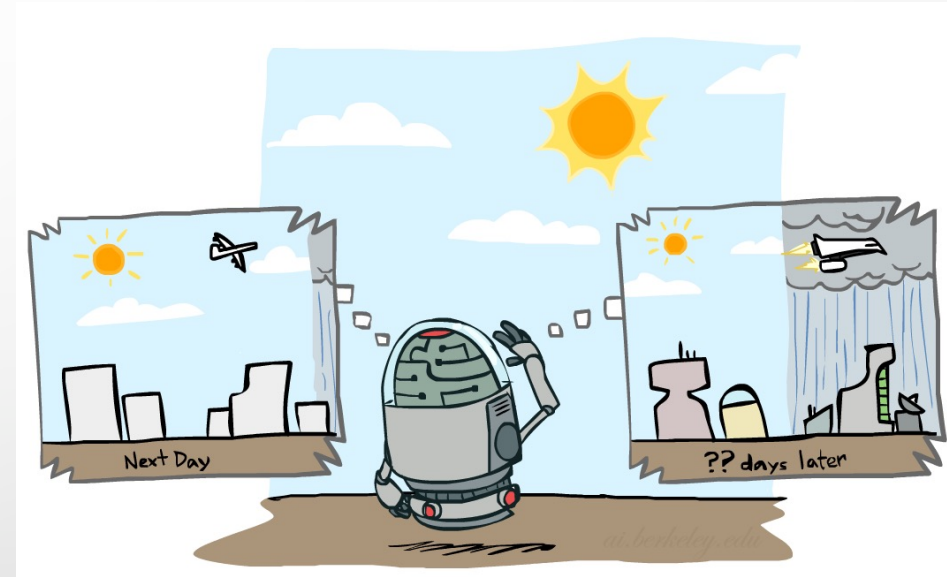
- Question: what's  $P(X)$  on some day  $t$ ?



We know:  $P(X_1), P(X_t|X_{t-1})$

$$\begin{aligned} P(x_t) &= \sum_{x_{t-1}} P(x_{t-1}, x_t) \\ &= \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1}) \end{aligned}$$

← *Forward simulation*



# Example Run of Mini-Forward Algorithm

- From initial observation of sun

$$\begin{array}{ccccccc} \left\langle \begin{array}{c} 1.0 \\ 0.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.9 \\ 0.1 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.84 \\ 0.16 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.804 \\ 0.196 \end{array} \right\rangle & \longrightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\ P(X_1) & P(X_2) & P(X_3) & P(X_4) & & P(X_\infty) \end{array}$$

- From initial observation of rain

$$\begin{array}{ccccccc} \left\langle \begin{array}{c} 0.0 \\ 1.0 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.3 \\ 0.7 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.48 \\ 0.52 \end{array} \right\rangle & \left\langle \begin{array}{c} 0.588 \\ 0.412 \end{array} \right\rangle & \longrightarrow & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\ P(X_1) & P(X_2) & P(X_3) & P(X_4) & & P(X_\infty) \end{array}$$

- From yet another initial distribution  $P(X_1)$ :

$$\begin{array}{ccc} \left\langle \begin{array}{c} p \\ 1-p \end{array} \right\rangle & \dots & \left\langle \begin{array}{c} 0.75 \\ 0.25 \end{array} \right\rangle \\ P(X_1) & & P(X_\infty) \end{array}$$

# Stationary Distributions

- For most chains:

- Influence of the initial distribution gets less and less over time.
- The distribution we end up in is independent of the initial distribution

- **Stationary distribution:**

- The distribution we end up with is called the **stationary distribution**  $P_\infty$  of the chain
- It satisfies

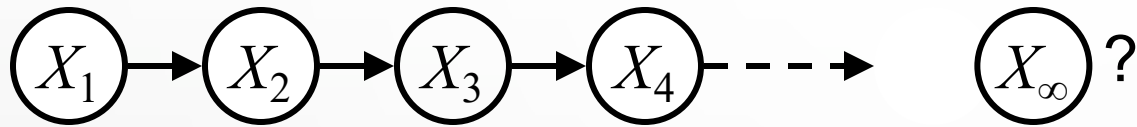
$$P_\infty(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_\infty(x)$$





# Example: Stationary Distributions

- Question: what's  $P(X)$  at time  $t = \text{infinity}$ ?



$$P_{\infty}(X) = P_{\infty+1}(X) = \sum_x P(X|x)P_{\infty}(x)$$

$$P_{\infty}(\text{sun}) = P(\text{sun}|\text{sun})P_{\infty}(\text{sun}) + P(\text{sun}|\text{rain})P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = P(\text{rain}|\text{sun})P_{\infty}(\text{sun}) + P(\text{rain}|\text{rain})P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{sun}) = 0.9P_{\infty}(\text{sun}) + 0.3P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{rain}) = 0.1P_{\infty}(\text{sun}) + 0.7P_{\infty}(\text{rain})$$

$$P_{\infty}(\text{sun}) = 3P_{\infty}(\text{rain})$$

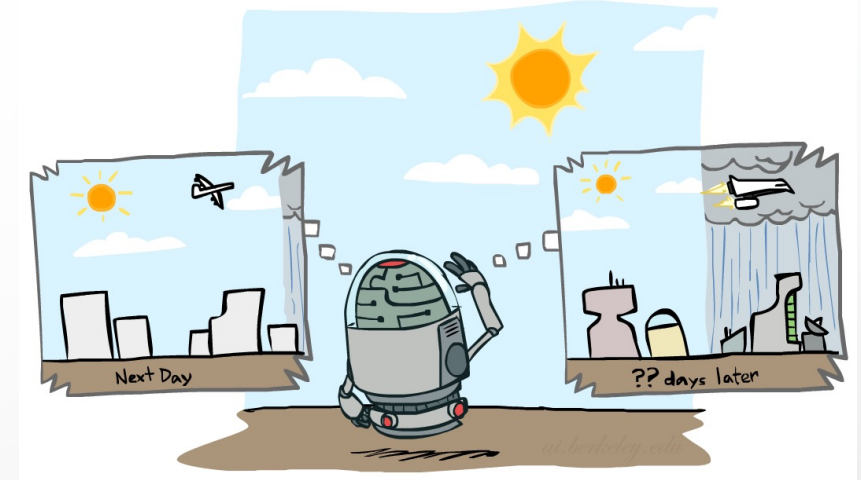
$$P_{\infty}(\text{rain}) = 1/3P_{\infty}(\text{sun})$$

Also:  $P_{\infty}(\text{sun}) + P_{\infty}(\text{rain}) = 1$



$$P_{\infty}(\text{sun}) = 3/4$$

$$P_{\infty}(\text{rain}) = 1/4$$

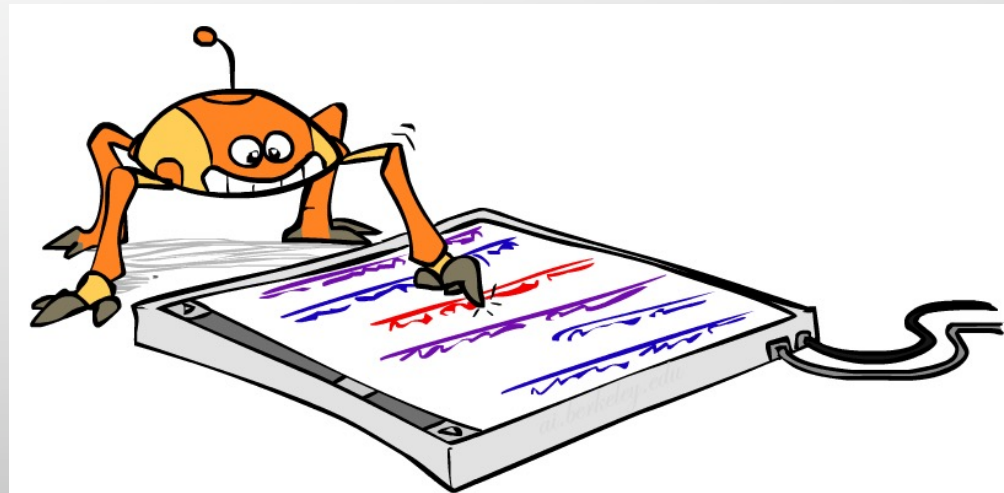
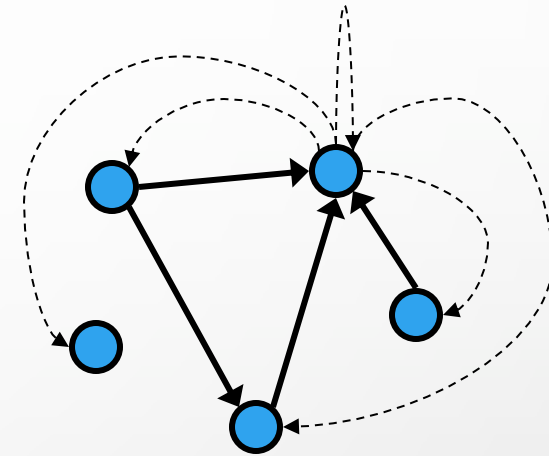


$X_{t-1}$	$X_t$	$P(X_t X_{t-1})$
sun	sun	0.9
sun	rain	0.1
rain	sun	0.3
rain	rain	0.7

- Alternatively: run simulation for a long (ideally infinite) time

# Application of Stationary Distribution: Web Link Analysis

- PageRank over a web graph
  - Each web page is a state
  - Initial distribution: uniform over pages
  - Transitions:
    - With prob.  $c$ , uniform jump to a random page (dotted lines, not all shown)
    - With prob.  $1-c$ , follow a random outlink (solid lines)
- Stationary distribution
  - Will spend more time on highly reachable pages
  - e.g. Many ways to get to the acrobat reader download page
  - Somewhat robust to link spam
  - Google 1.0 returned the set of pages containing all your keywords in decreasing rank, now all search engines use link analysis along with many other factors (rank actually getting less important over time)





# Application of Stationary Distributions: Gibbs Sampling

- Each joint instantiation over all hidden and query variables is a state:  $\{X_1, \dots, X_n\} = H \cup Q$

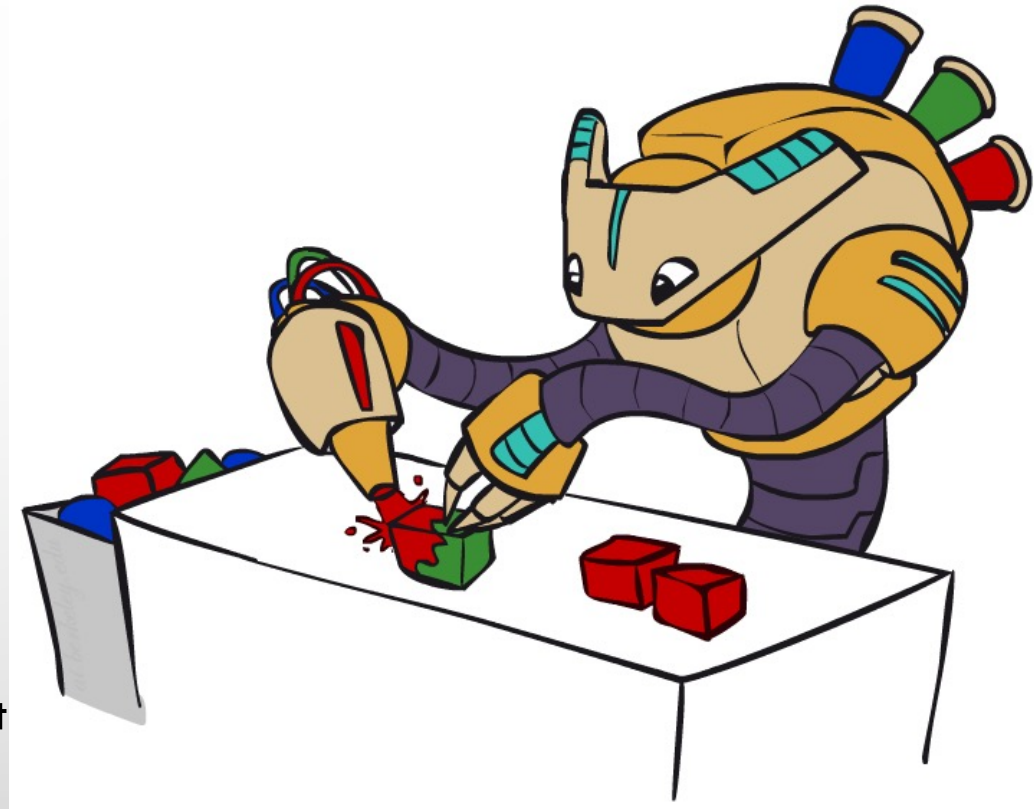
- Transitions:

- Resample variable  $x_i$  according to

$$p(X_i \mid X_1, X_2, \dots, X_{i-1}, X_{i+1}, \dots, X_n, E_1, \dots, E_m)$$

- Stationary distribution:

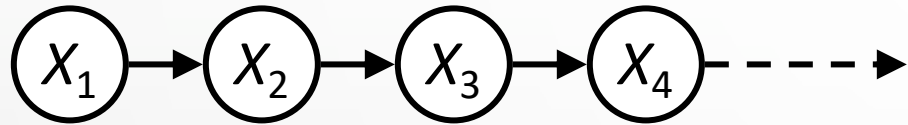
- Conditional distribution  $P(X_1, X_2, \dots, X_n \mid E_1, \dots, E_m)$
- Means that when running Gibbs sampling long enough we get a sample from the desired distribution
- Requires some proof to show this is true!



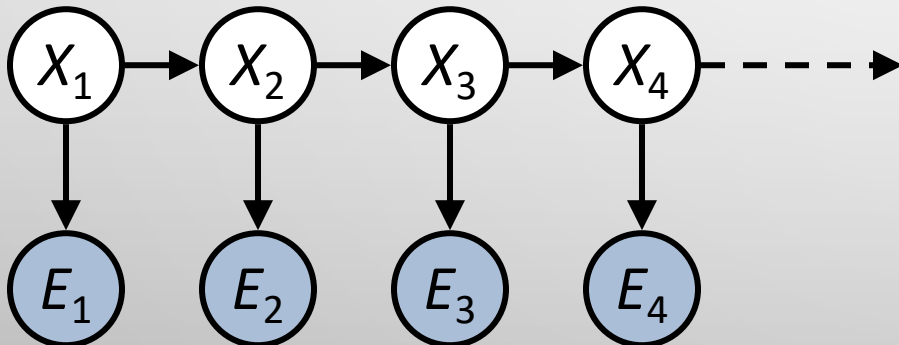
# Hidden Markov Models

# Hidden Markov Models

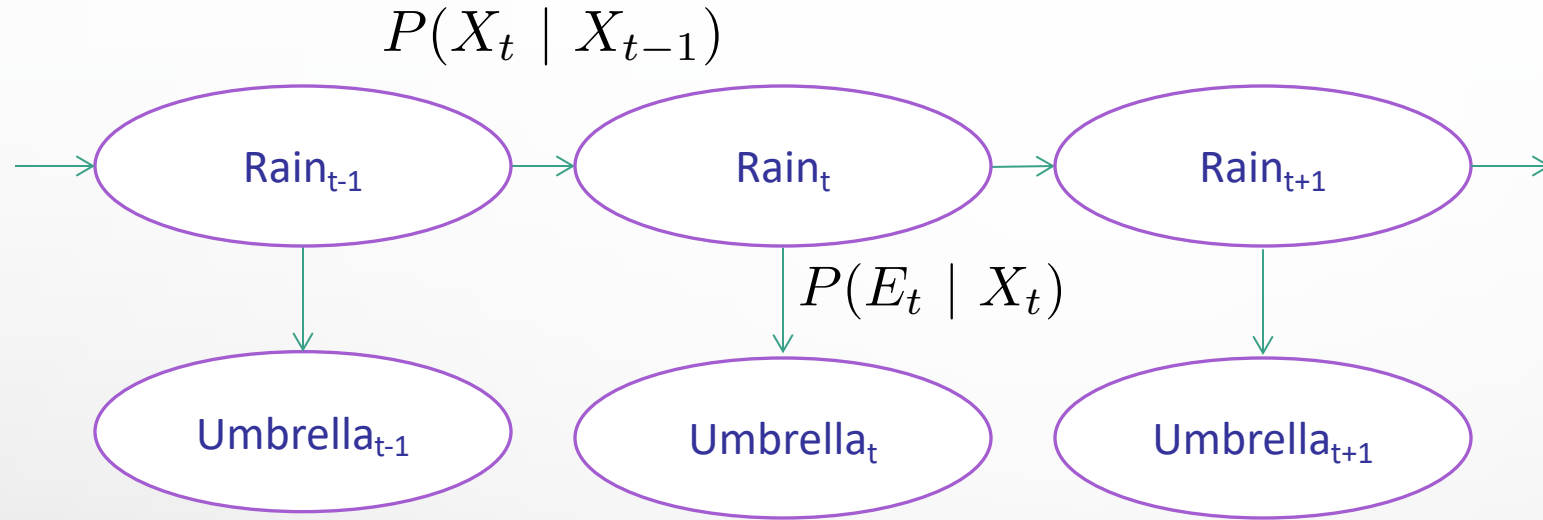
- Markov chains not so useful for most agents



- Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states  $X$
  - You observe outputs (effects) at each time step



# Example: Weather HMM



- An HMM is defined by:

- Initial distribution:  $P(X_1)$

- Transitions:  $P(X_t | X_{t-1})$

- Emissions:  $P(E_t | X_t)$

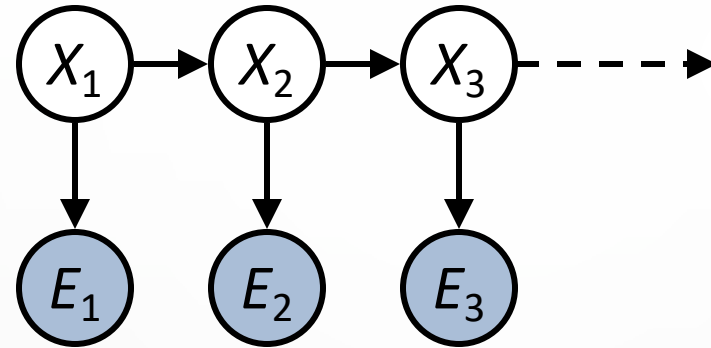
Transmissions

$R_t$	$R_{t+1}$	$P(R_{t+1}   R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

Emissions

$R_t$	$U_t$	$P(U_t   R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

# Joint Distribution of an HMM



- Joint distribution:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

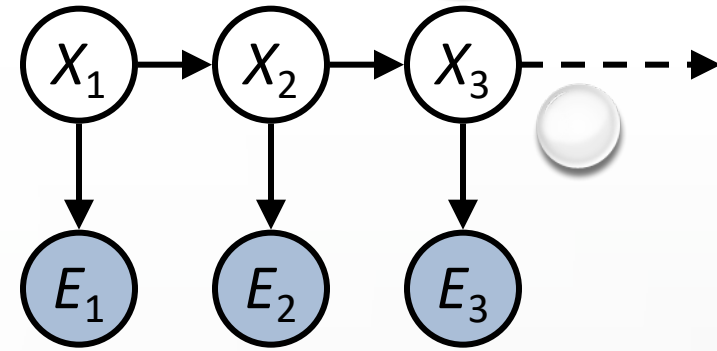
- More generally:

$$P(X_1, E_1, \dots, X_T, E_T) = P(X_1)P(E_1|X_1) \prod_{t=2}^T P(X_t|X_{t-1})P(E_t|X_t)$$

- Questions to be resolved:

- Does this indeed define a joint distribution?
- Can every joint distribution be factored this way, or are we making some assumptions about the joint distribution by using this factorization?

# Chain Rule and HMMs



- From the chain rule, *every* joint distribution over  $X_1, E_1, X_2, E_2, X_3, E_3$  can be written as:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1, E_1)P(E_2|X_1, E_1, X_2) \\ P(X_3|X_1, E_1, X_2, E_2)P(E_3|X_1, E_1, X_2, E_2, X_3)$$

- Assuming that

$$X_2 \perp\!\!\!\perp E_1 \mid X_1, \quad E_2 \perp\!\!\!\perp X_1, E_1 \mid X_2, \quad X_3 \perp\!\!\!\perp X_1, E_1, E_2 \mid X_2, \quad E_3 \perp\!\!\!\perp X_1, E_1, X_2, E_2 \mid X_3$$

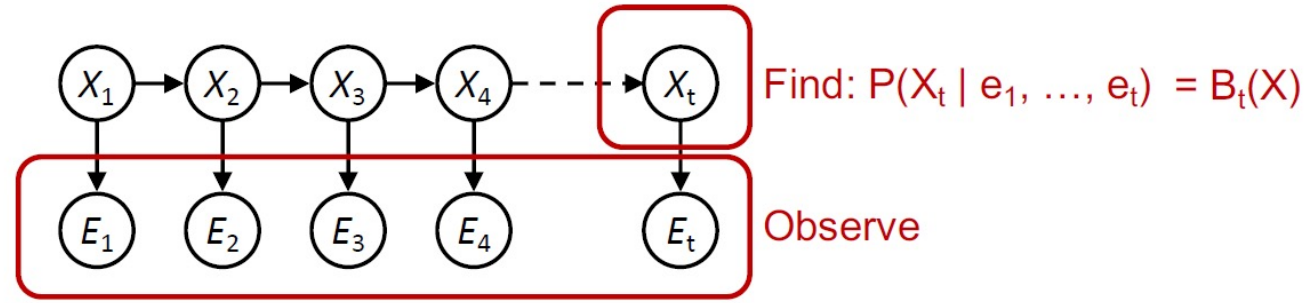
Gives us the expression posited on the previous slide:

$$P(X_1, E_1, X_2, E_2, X_3, E_3) = P(X_1)P(E_1|X_1)P(X_2|X_1)P(E_2|X_2)P(X_3|X_2)P(E_3|X_3)$$

# Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous)
  - States are positions on a map (continuous)

# Filtering / Monitoring

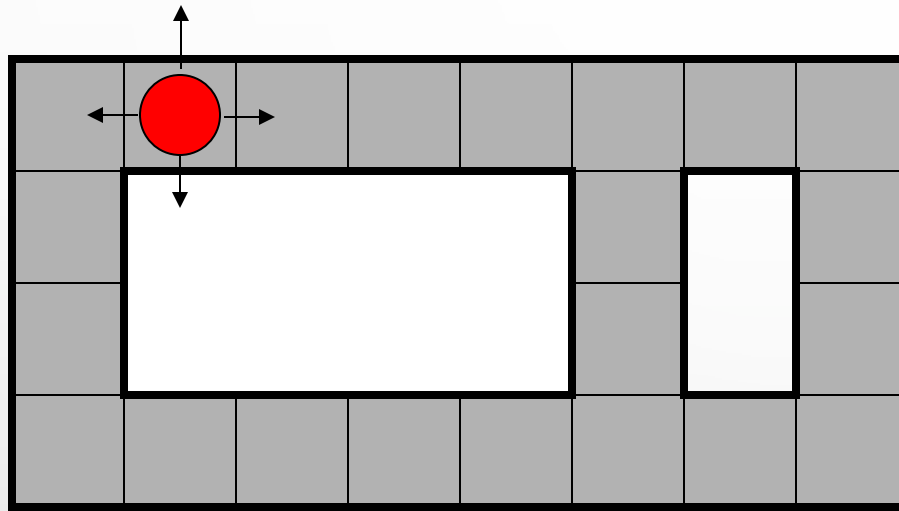


- Filtering, or monitoring, is the task of tracking the distribution  $B_t(x) = P(X_t | E_1, \dots, E_t)$  (the belief state) over time
- We start with  $B_0(x)$  in an initial setting, usually uniform
- As time passes, or we get observations, we update  $B(x)$
- The Kalman filter was invented in the 60's and first implemented as a method of trajectory estimation for the Apollo program



# Example: Robot Localization

Example from  
Michael Pfeiffer

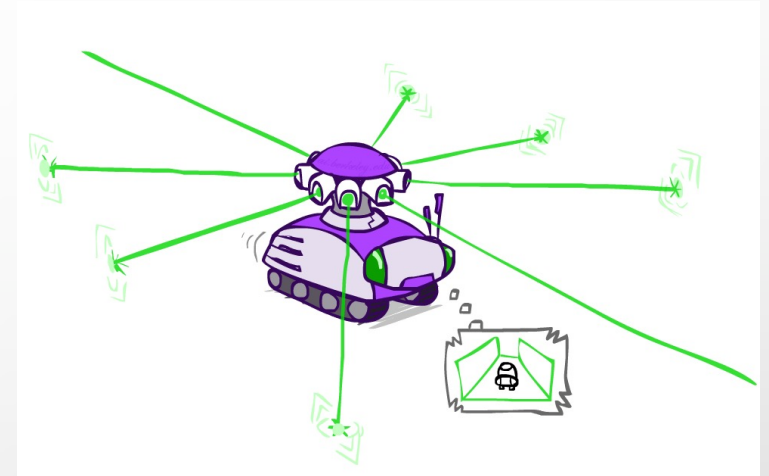


Prob

0

1

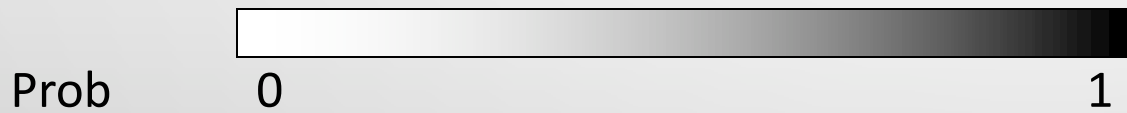
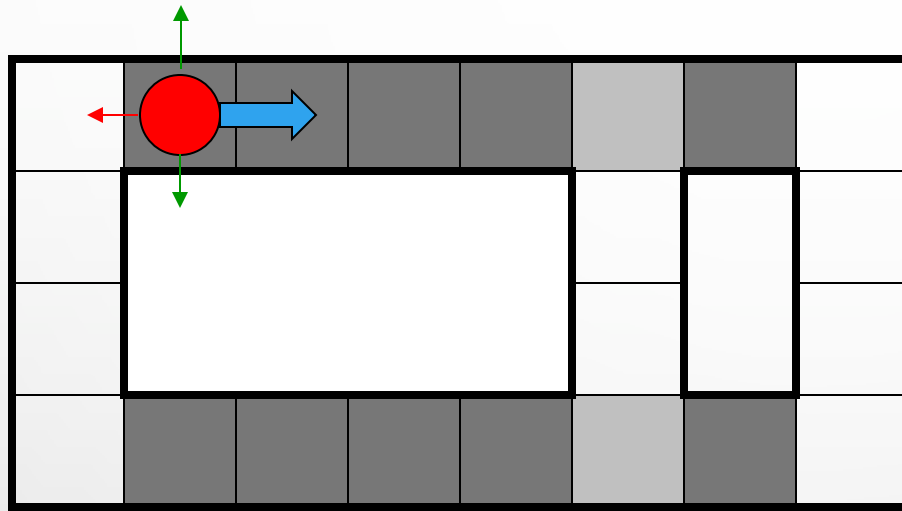
t=0



Sensor model: can read in which directions there is a wall, never more than 1 mistake

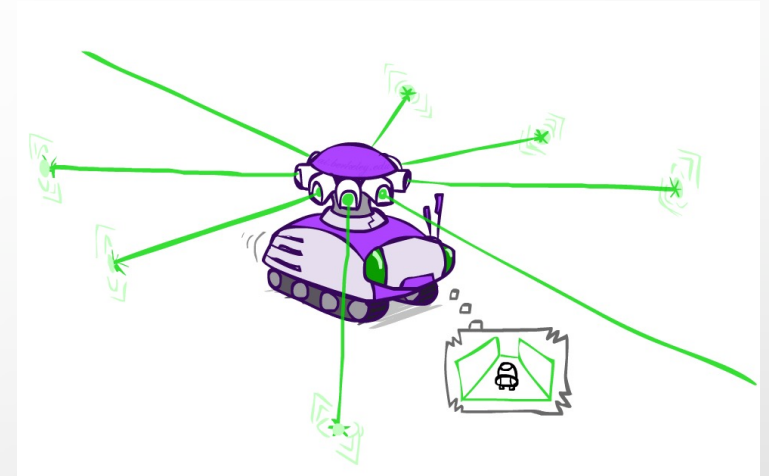
Motion model: may not execute action with small prob.

# Example: Robot Localization

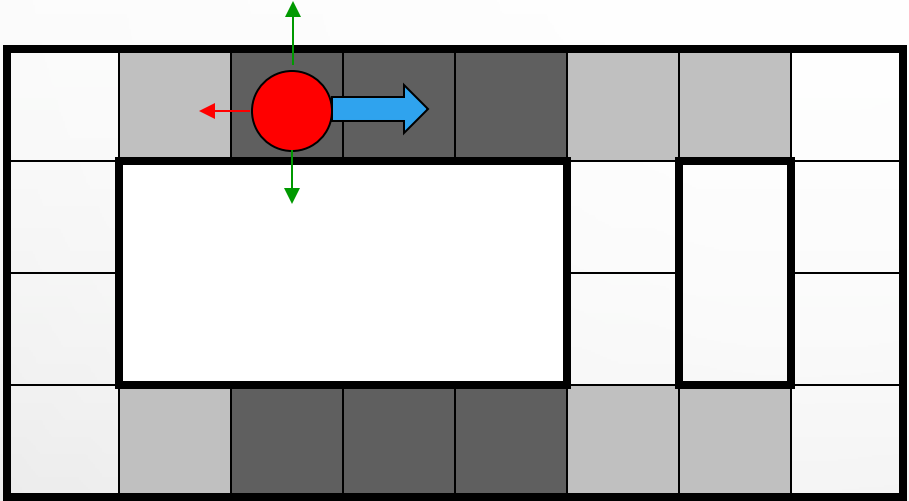


t=1

Lighter grey: was possible to get the reading, but less likely b/c required 1 mistake



# Example: Robot Localization

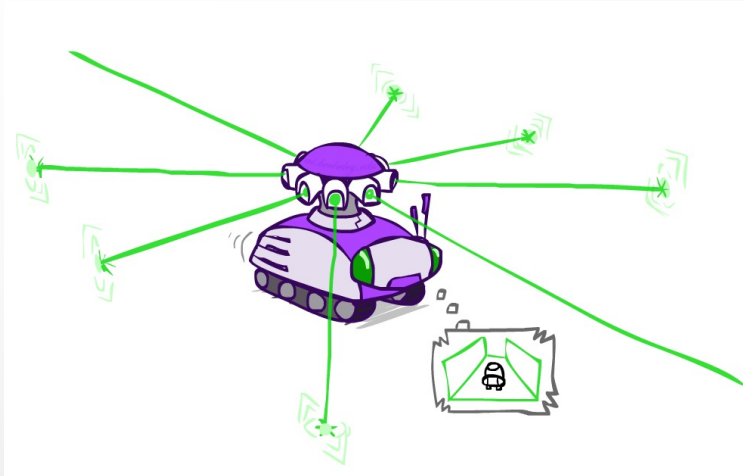


Prob

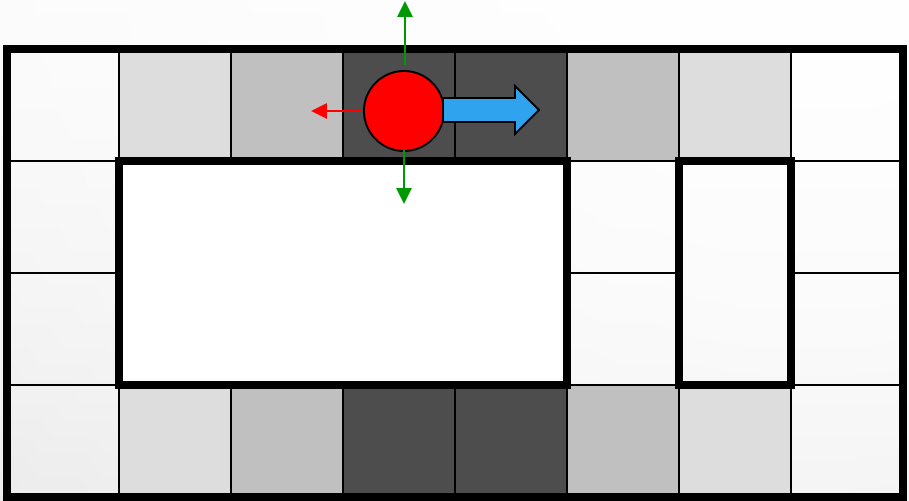
0

1

t=2



# Example: Robot Localization

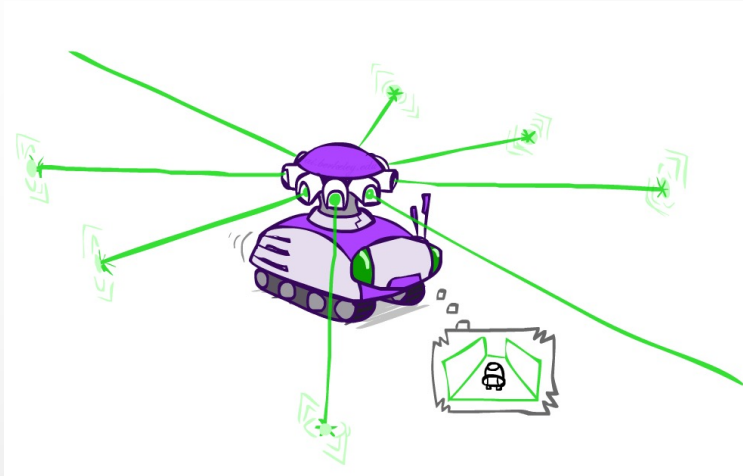


Prob

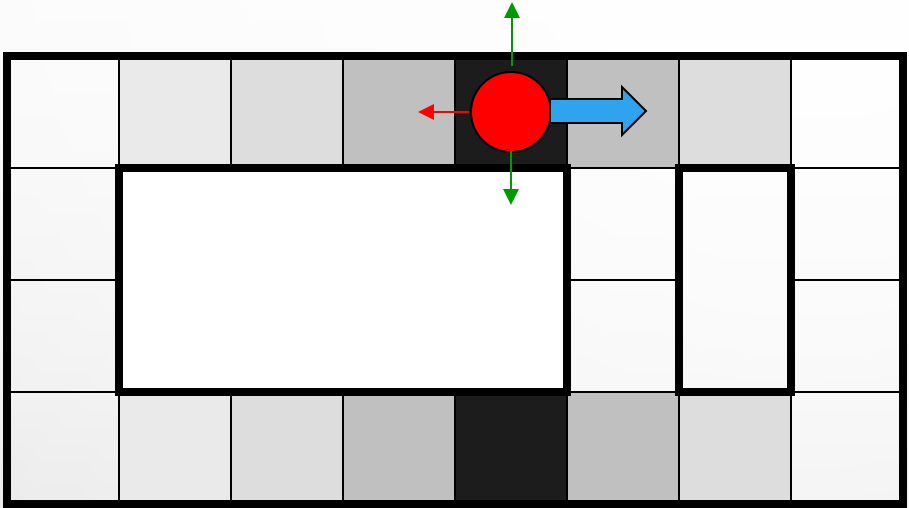
0

1

t=3



# Example: Robot Localization

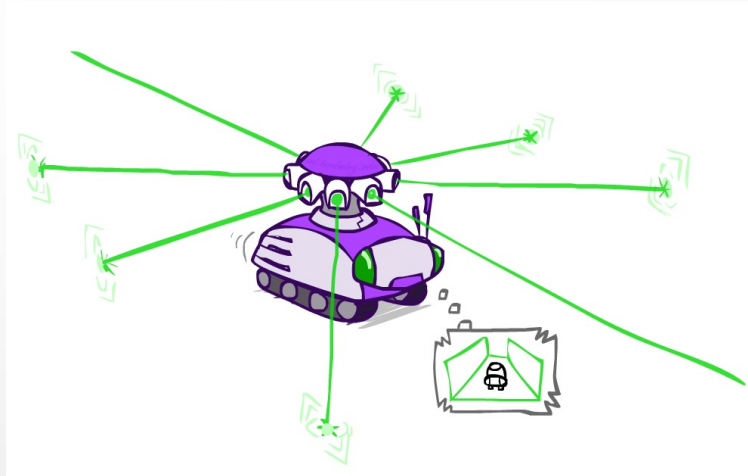


Prob

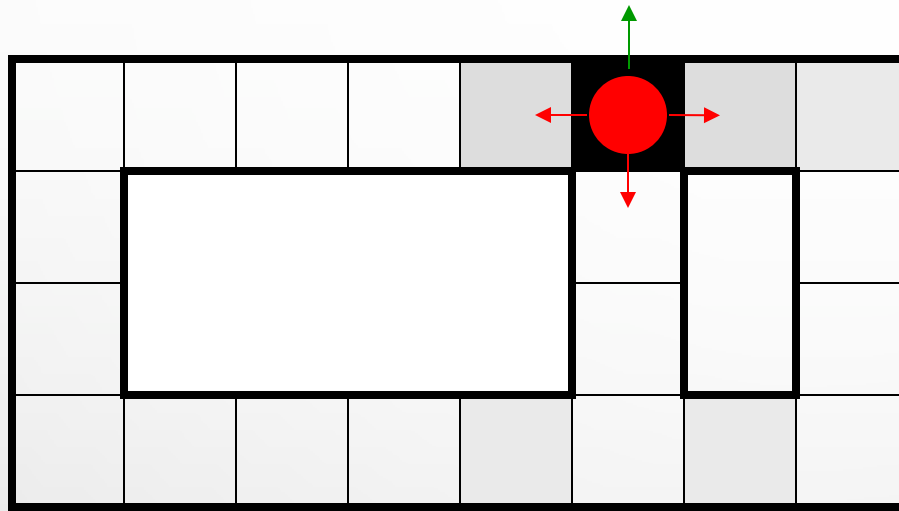
0

1

$t=4$



# Example: Robot Localization

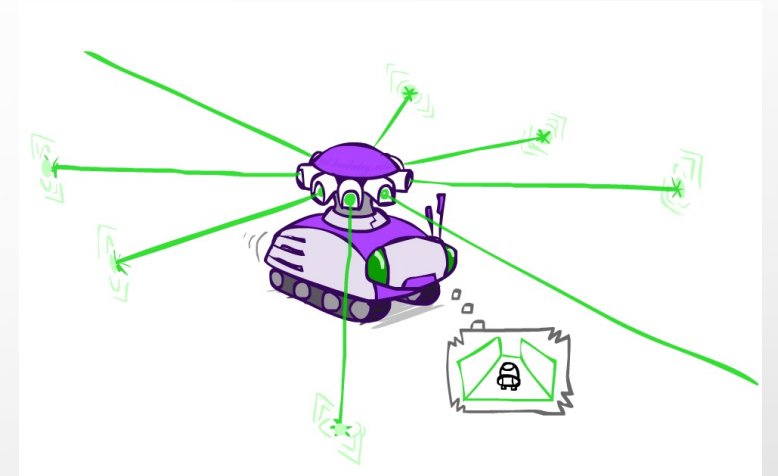


Prob

0

1

t=5



# Inference: Find State Given Evidence

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t | e_{1:t})$$

- Idea: start with  $P(X_1)$  and derive  $B_t$  in terms of  $B_{t-1}$

# Inference: Base Cases

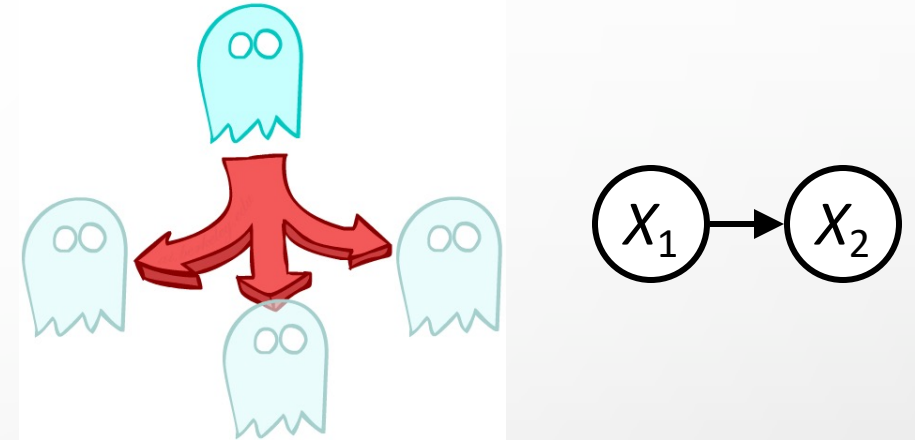
Observation:



$$P(X_1|e_1)$$

$$\begin{aligned} P(x_1|e_1) &= P(x_1, e_1)/P(e_1) \\ &\propto_{X_1} P(x_1, e_1) \\ &= P(x_1)P(e_1|x_1) \end{aligned}$$

Passage of Time:



$$P(X_2)$$

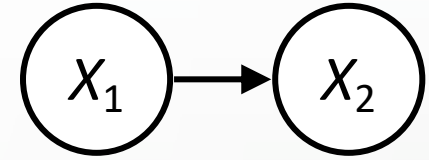
$$\begin{aligned} P(x_2) &= \sum_{x_1} P(x_1, x_2) \\ &= \sum_{x_1} P(x_1)P(x_2|x_1) \end{aligned}$$



# Passage of Time

- Assume we have current belief  $P(X \mid \text{evidence to date})$

$$B(X_t) = P(X_t | e_{1:t})$$



- then, after one time step passes:

$$\begin{aligned} P(X_{t+1} | e_{1:t}) &= \sum_{x_t} P(X_{t+1}, x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t, e_{1:t}) P(x_t | e_{1:t}) \\ &= \sum_{x_t} P(X_{t+1} | x_t) P(x_t | e_{1:t}) \end{aligned}$$

- Or compactly:

$$B'(X_{t+1}) = \sum_{x_t} P(X' | x_t) B(x_t)$$

- Basic idea: beliefs get “pushed” through the transitions
  - With the “B” notation, we have to be careful about what time step  $t$  the belief is about, and what evidence it includes

# Example: Passage of Time

- As time passes, uncertainty “accumulates”

(Transition model: ghosts usually go clockwise)

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	1.00	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

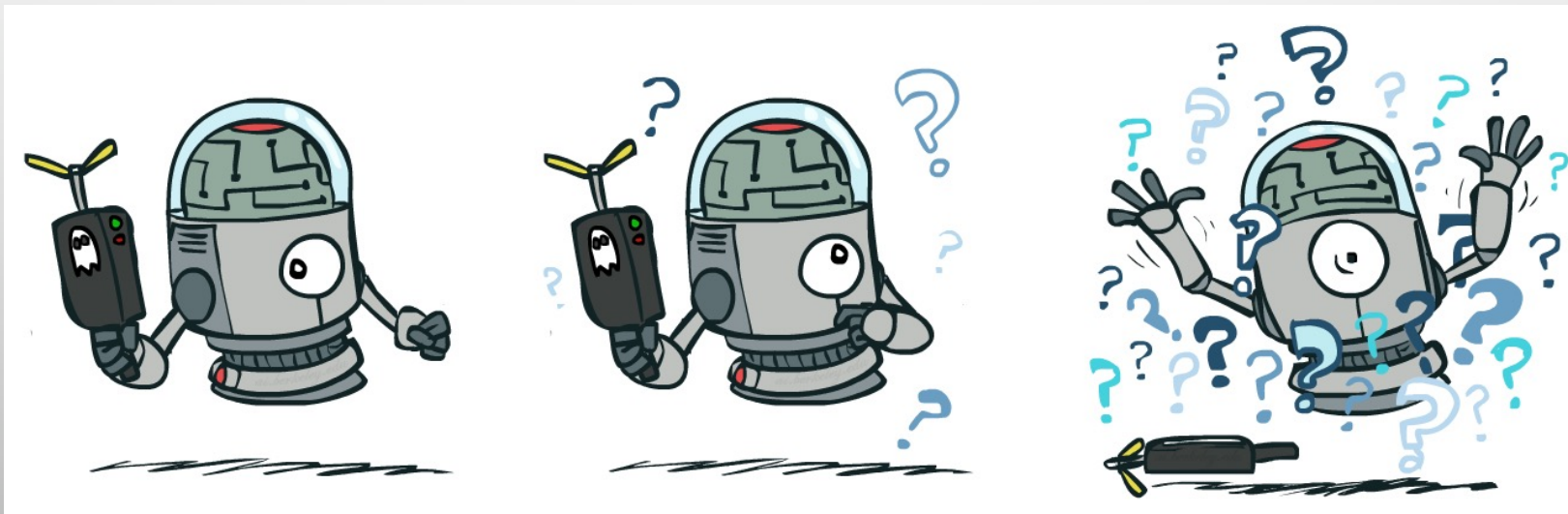
t = 1

<0.01	<0.01	<0.01	<0.01	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01
<0.01	0.76	0.06	0.06	<0.01	<0.01
<0.01	<0.01	0.06	<0.01	<0.01	<0.01

t = 2

0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

t = 5



# Observation

- Assume we have current belief  $P(X \mid \text{previous evidence})$ :

$$B'(X_{t+1}) = P(X_{t+1} | e_{1:t})$$

- Then, after evidence comes in:

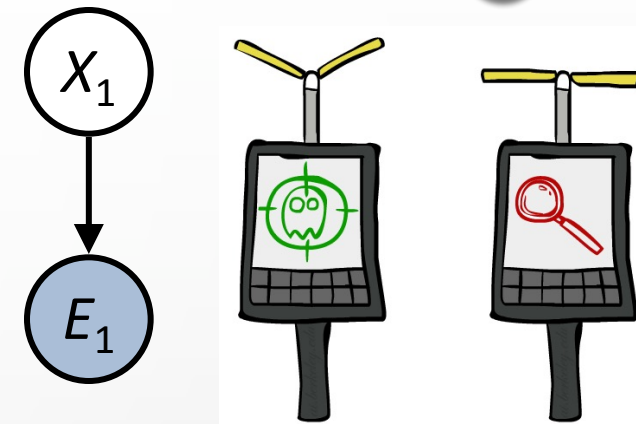
$$\begin{aligned} P(X_{t+1} | e_{1:t+1}) &= P(X_{t+1}, e_{t+1} | e_{1:t}) / P(e_{t+1} | e_{1:t}) \\ &\propto_{X_{t+1}} P(X_{t+1}, e_{t+1} | e_{1:t}) \end{aligned}$$

$$= P(e_{t+1} | e_{1:t}, X_{t+1}) P(X_{t+1} | e_{1:t})$$

$$= P(e_{t+1} | X_{t+1}) P(X_{t+1} | e_{1:t})$$

- Or, compactly:

$$B(X_{t+1}) \propto_{X_{t+1}} P(e_{t+1} | X_{t+1}) B'(X_{t+1})$$



- Basic idea: beliefs “reweighted” by likelihood of evidence
- Unlike passage of time, we have to renormalize

# Example: Observation

- As we get observations, beliefs get reweighted, uncertainty “decreases”

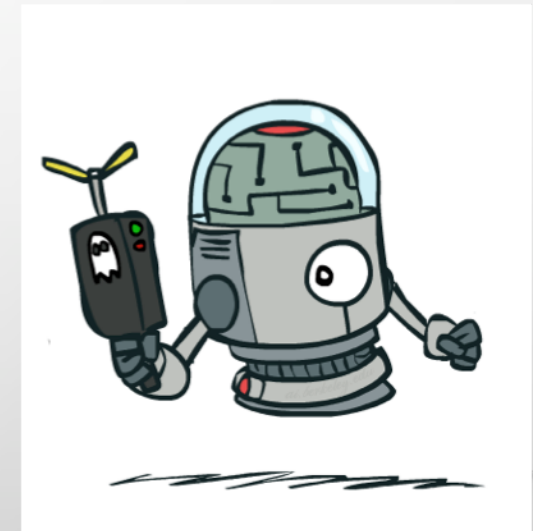
0.05	0.01	0.05	<0.01	<0.01	<0.01
0.02	0.14	0.11	0.35	<0.01	<0.01
0.07	0.03	0.05	<0.01	0.03	<0.01
0.03	0.03	<0.01	<0.01	<0.01	<0.01

Before observation

<0.01	<0.01	<0.01	<0.01	0.02	<0.01
<0.01	<0.01	<0.01	0.83	0.02	<0.01
<0.01	<0.01	0.11	<0.01	<0.01	<0.01
<0.01	<0.01	<0.01	<0.01	<0.01	<0.01

After observation

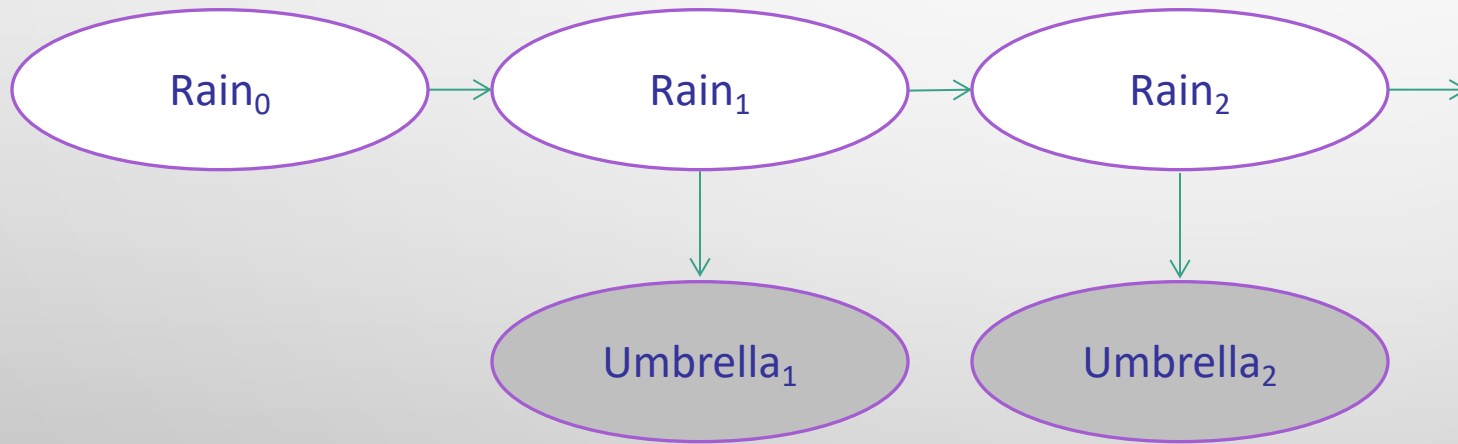
$$B(X) \propto P(e|X)B'(X)$$



# Example: Weather HMM



$$\begin{array}{l}
 B(+r) = 0.5 \\
 B(-r) = 0.5
 \end{array}
 \begin{array}{l}
 \nearrow \\
 \downarrow
 \end{array}
 \begin{array}{l}
 B'(+r) = 0.5 \\
 B'(-r) = 0.5
 \end{array}
 \begin{array}{l}
 \nearrow \\
 \downarrow
 \end{array}
 \begin{array}{l}
 B(+r) = 0.818 \\
 B(-r) = 0.182
 \end{array}
 \begin{array}{l}
 \nearrow \\
 \downarrow
 \end{array}
 \begin{array}{l}
 B'(+r) = 0.627 \\
 B'(-r) = 0.373
 \end{array}
 \begin{array}{l}
 \nearrow \\
 \downarrow
 \end{array}
 \begin{array}{l}
 B(+r) = 0.883 \\
 B(-r) = 0.117
 \end{array}$$



$R_t$	$R_{t+1}$	$P(R_{t+1} R_t)$
+r	+r	0.7
+r	-r	0.3
-r	+r	0.3
-r	-r	0.7

$R_t$	$U_t$	$P(U_t R_t)$
+r	+u	0.9
+r	-u	0.1
-r	+u	0.2
-r	-u	0.8

# The Forward Algorithm

- We are given evidence at each time and want to know

$$B_t(X) = P(X_t | e_{1:t})$$

- We can derive the following updates

$$P(x_t | e_{1:t}) \propto_X P(x_t, e_{1:t})$$

We can normalize as we go if we want to have  $P(x|e)$  at each time step, or just once at the end...

$$= \sum_{x_{t-1}} P(x_{t-1}, x_t, e_{1:t})$$

$$= \sum_{x_{t-1}} P(x_{t-1}, e_{1:t-1}) P(x_t | x_{t-1}) P(e_t | x_t)$$

$$= P(e_t | x_t) \sum_{x_{t-1}} P(x_t | x_{t-1}) P(x_{t-1}, e_{1:t-1})$$



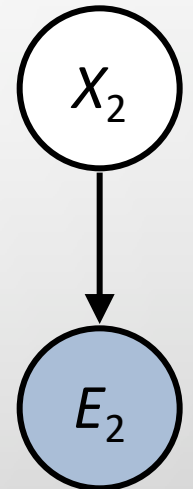
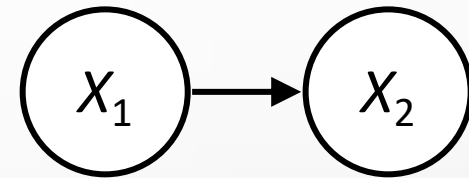
# Online Belief Updates

- Every time step, we start with current  $P(X \mid \text{evidence})$

- We update for time:  $P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$

- We update for evidence:  $P(x_t | e_{1:t}) \propto_X P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$

- The forward algorithm does both at once (and doesn't normalize)



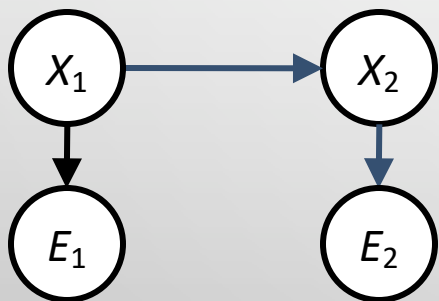
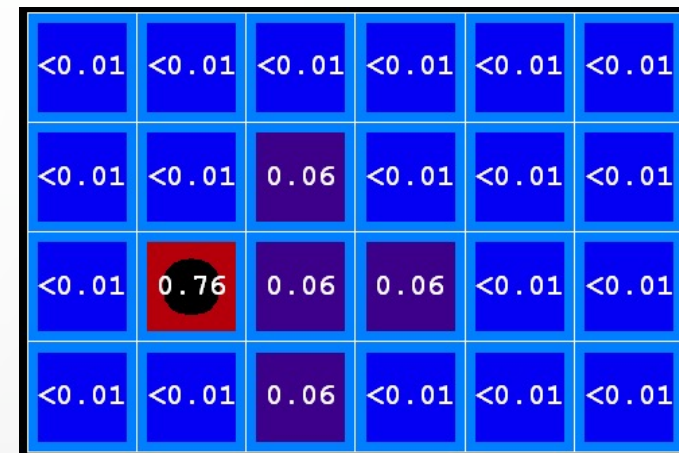
# Recap: Filtering

**Elapse time:** compute  $P(X_t | e_{1:t-1})$

$$P(x_t | e_{1:t-1}) = \sum_{x_{t-1}} P(x_{t-1} | e_{1:t-1}) \cdot P(x_t | x_{t-1})$$

**Observe:** compute  $P(X_t | e_{1:t})$

$$P(x_t | e_{1:t}) \propto P(x_t | e_{1:t-1}) \cdot P(e_t | x_t)$$



**Belief:**  $\langle P(\text{rain}), P(\text{sun}) \rangle$

$P(X_1)$        $\langle 0.5, 0.5 \rangle$       *Prior on  $X_1$*

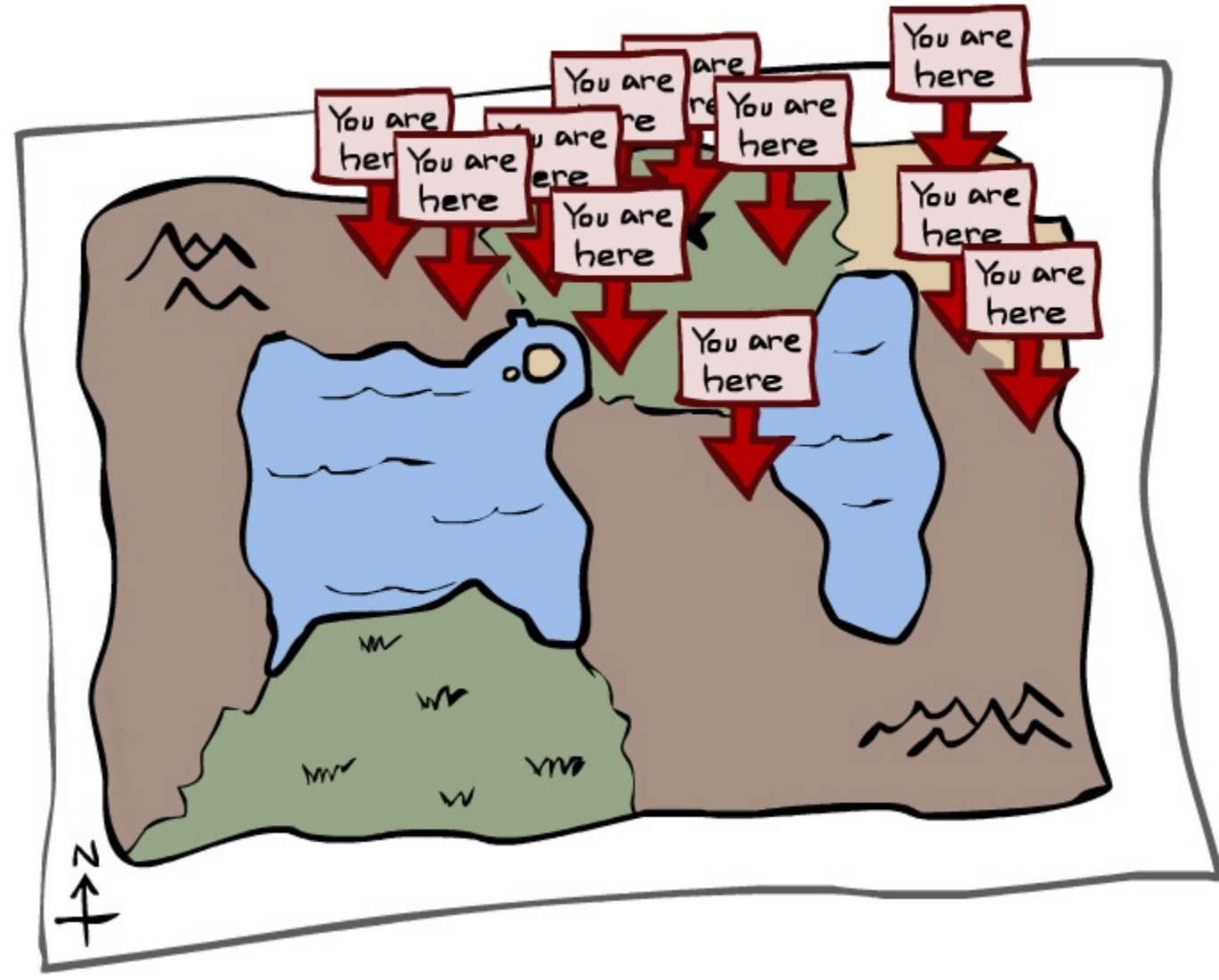
$P(X_1 | E_1 = \text{umbrella})$        $\langle 0.82, 0.18 \rangle$       *Observe*

$P(X_2 | E_1 = \text{umbrella})$        $\langle 0.63, 0.37 \rangle$       *Elapse time*

$P(X_2 | E_1 = \text{umb}, E_2 = \text{umb})$        $\langle 0.88, 0.12 \rangle$       *Observe*



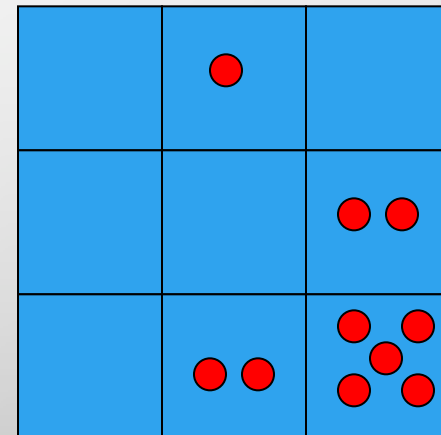
# Particle Filtering



# Particle Filtering

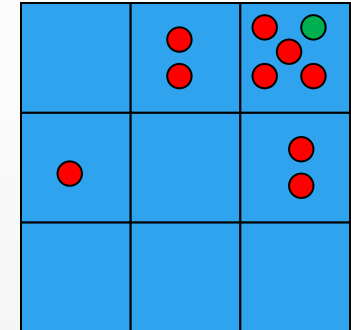
- Filtering: approximate solution
- Sometimes  $|X|$  is too big to use exact inference
  - $|X|$  may be too big to even store  $B(X)$
  - E.g.  $X$  is continuous
- Solution: approximate inference
  - Track samples of  $X$ , not all values
  - Samples are called particles
  - Time per step is linear in the number of samples
  - But: number needed may be large
  - In memory: list of particles, not states
- This is how robot localization works in practice
- Particle is just new name for sample

0.0	0.1	0.0
0.0	0.0	0.2
0.0	0.2	0.5



# Representation: Particles

- Our representation of  $P(X)$  is now a list of  $N$  particles (samples)
  - Generally,  $N \ll |X|$
  - Storing map from  $X$  to counts would defeat the point
- $P(x)$  approximated by number of particles with value  $x$ 
  - So, many  $x$  may have  $p(x) = 0$ !
  - More particles, more accuracy
- For now, all particles have a weight of 1



Particles:

(3,3)  
(2,3)  
(3,3)  
(3,2)  
(3,3)  
(3,2)  
(1,2)  
(3,3)  
(3,3)  
(2,3)

# Particle Filtering: Elapse Time

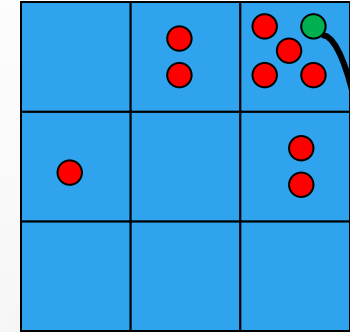
- Each particle is moved by sampling its next position from the transition model

$$x' = \text{sample}(P(X'|x))$$

- This is like prior sampling – samples' frequencies reflect the transition probabilities
- Here, most samples move clockwise, but some move in another direction or stay in place
- This captures the passage of time
  - If enough samples, close to exact values before and after (consistent)

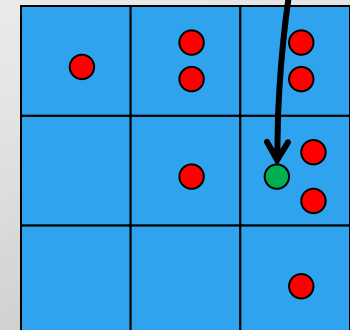
Particles:

(3,3)  
(2,3)  
(3,3)  
(3,2)  
(3,3)  
(3,2)  
(1,2)  
(3,3)  
(3,3)  
(2,3)



Particles:

(3,2)  
(2,3)  
(3,2)  
(3,1)  
(3,3)  
(3,2)  
(1,3)  
(2,3)  
(3,2)  
(2,2)



# Particle Filtering: Observe

- Slightly trickier:

- Don't sample observation, fix it
- Similar to likelihood weighting, downweight samples based on the evidence

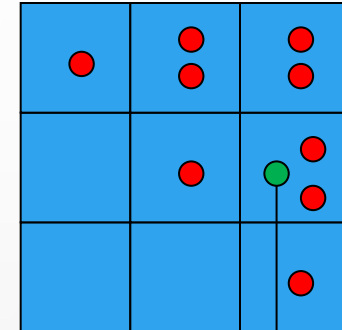
$$w(x) = P(e|x)$$

$$B(X) \propto P(e|X)B'(X)$$

- As before, the probabilities don't sum to one, since all have been down weighted (in fact they now sum to (N times) an approximation of P(e))

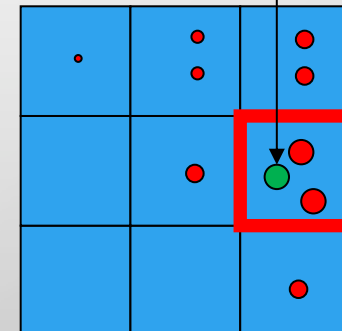
Particles:

(3,2)  
(2,3)  
(3,2)  
(3,1)  
(3,3)  
(3,2)  
(1,3)  
(2,3)  
(3,2)  
(2,2)



Particles:

(3,2) w=.9  
(2,3) w=.2  
(3,2) w=.9  
(3,1) w=.4  
(3,3) w=.4  
(3,2) w=.9  
(1,3) w=.1  
(2,3) w=.2  
(3,2) w=.9  
(2,2) w=.4



# Particle Filtering: Resample

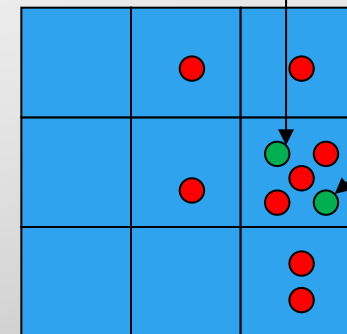
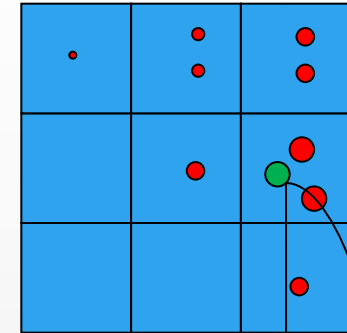
- Rather than tracking weighted samples, we resample
- $N$  times, we choose from our weighted sample distribution (i.e. draw with replacement)
- This is equivalent to renormalizing the distribution
- Now the update is complete for this time step, continue with the next one

Particles:

(3,2)  $w=.9$   
(2,3)  $w=.2$   
(3,2)  $w=.9$   
(3,1)  $w=.4$   
(3,3)  $w=.4$   
(3,2)  $w=.9$   
(1,3)  $w=.1$   
(2,3)  $w=.2$   
(3,2)  $w=.9$   
(2,2)  $w=.4$

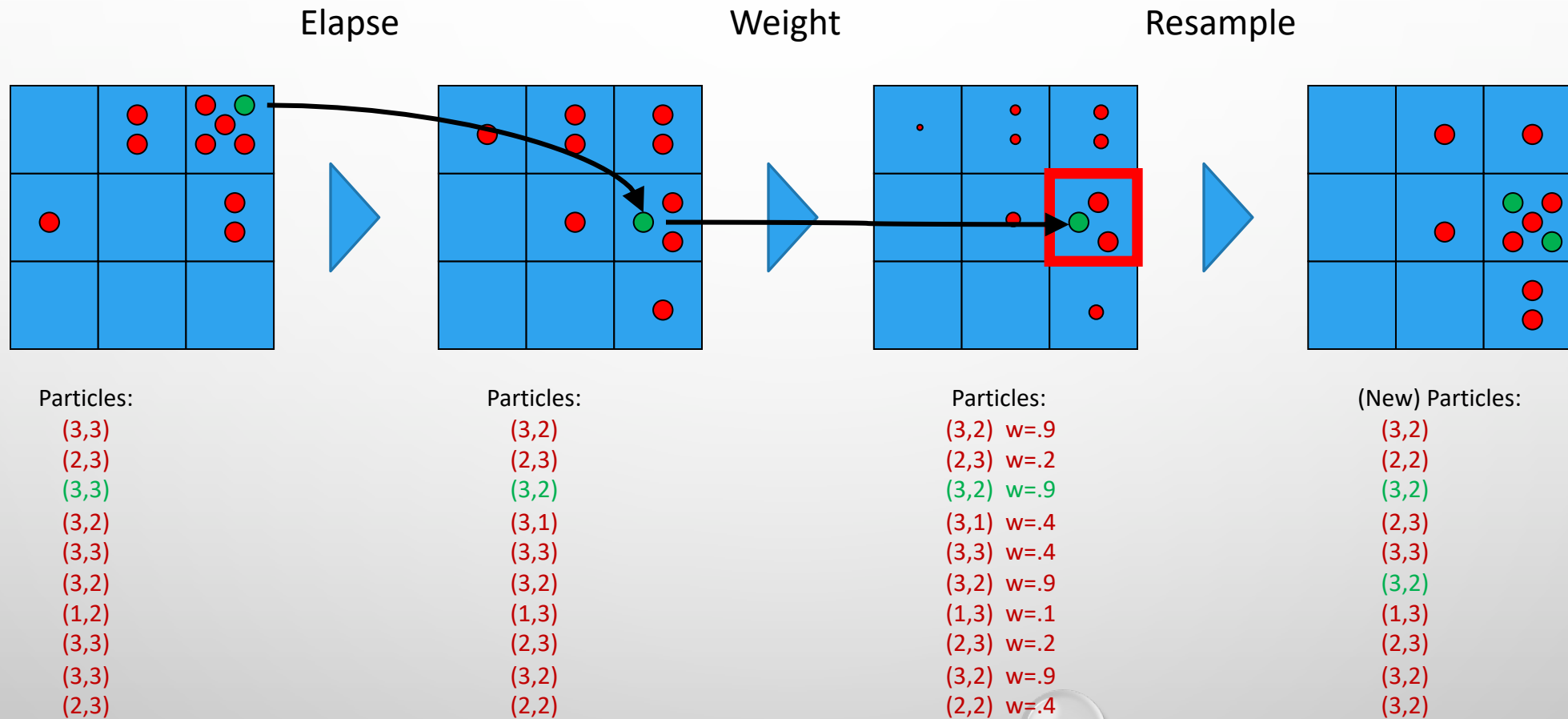
(New) Particles:

(3,2)  
(2,2)  
(3,2)  
(2,3)  
(3,3)  
(3,2)  
(1,3)  
(2,3)  
(3,2)  
(3,2)



# Recap: Particle Filtering

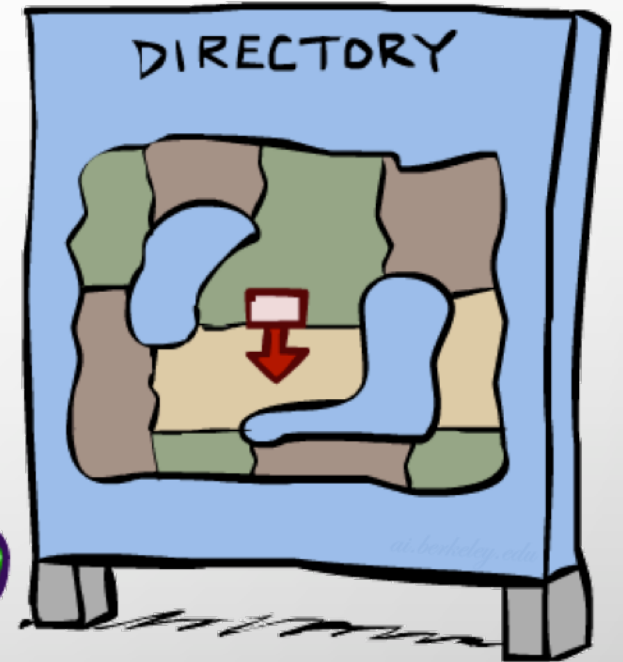
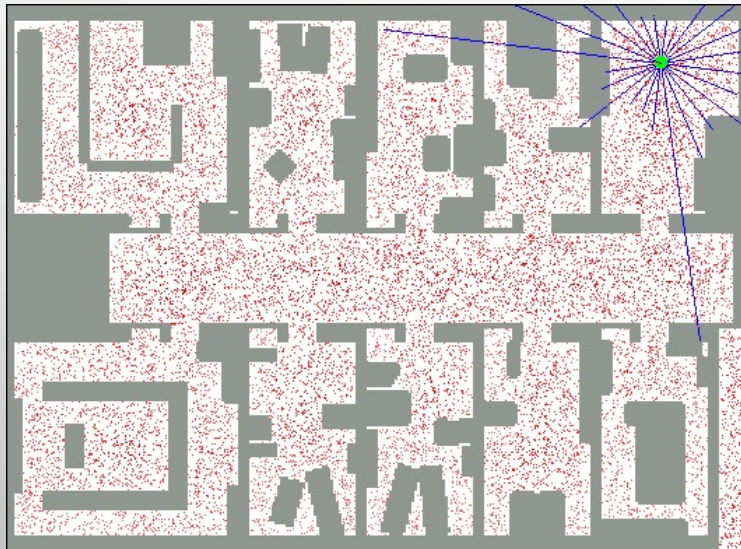
- Particles: track samples of states rather than an explicit distribution





# Robot Localization

- In robot localization:
  - We know the map, but not the robot's position
  - Observations may be vectors of range finder readings
  - State space and readings are typically continuous (works basically like a very fine grid) and so we cannot store  $B(X)$
  - Particle filtering is a main technique

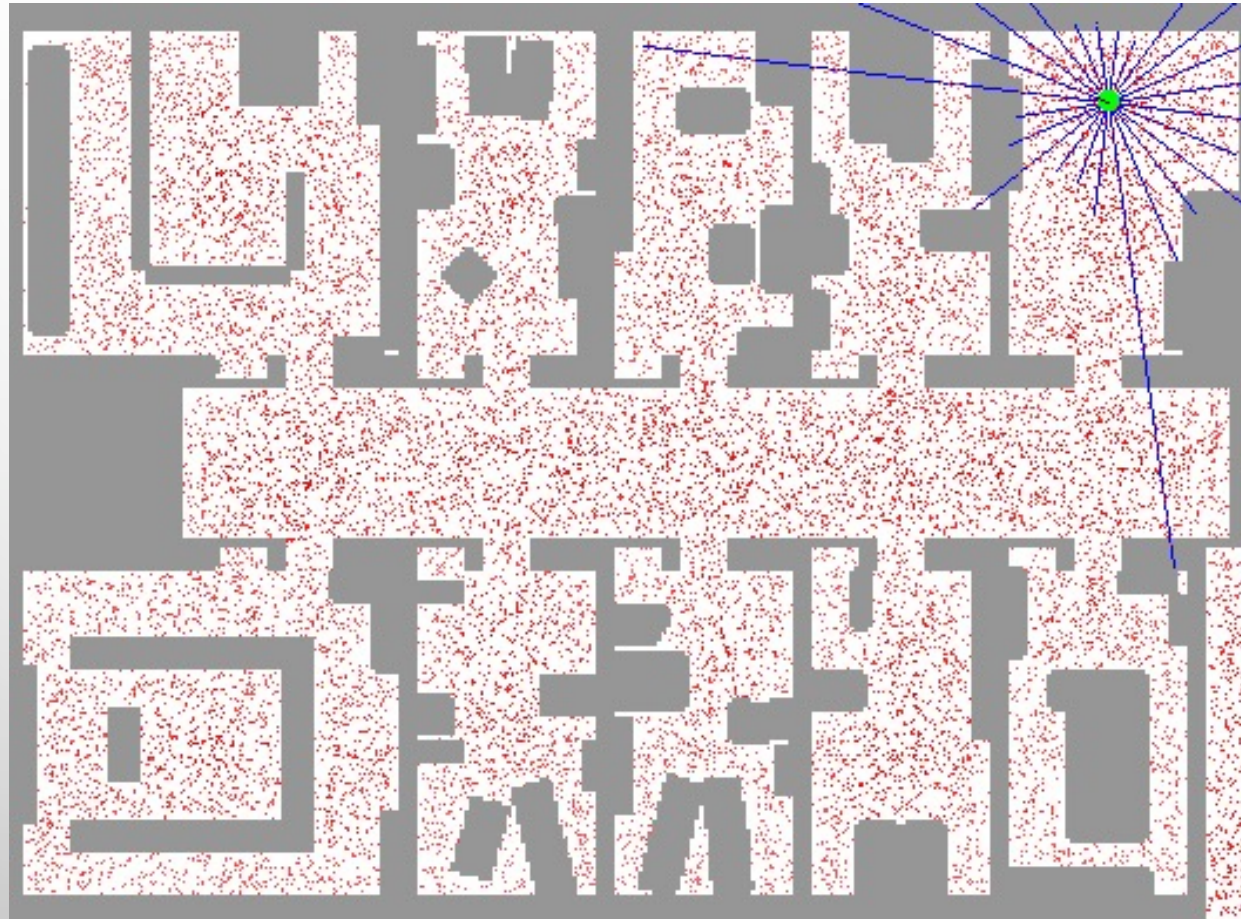




## Particle Filter Localization (Sonar)

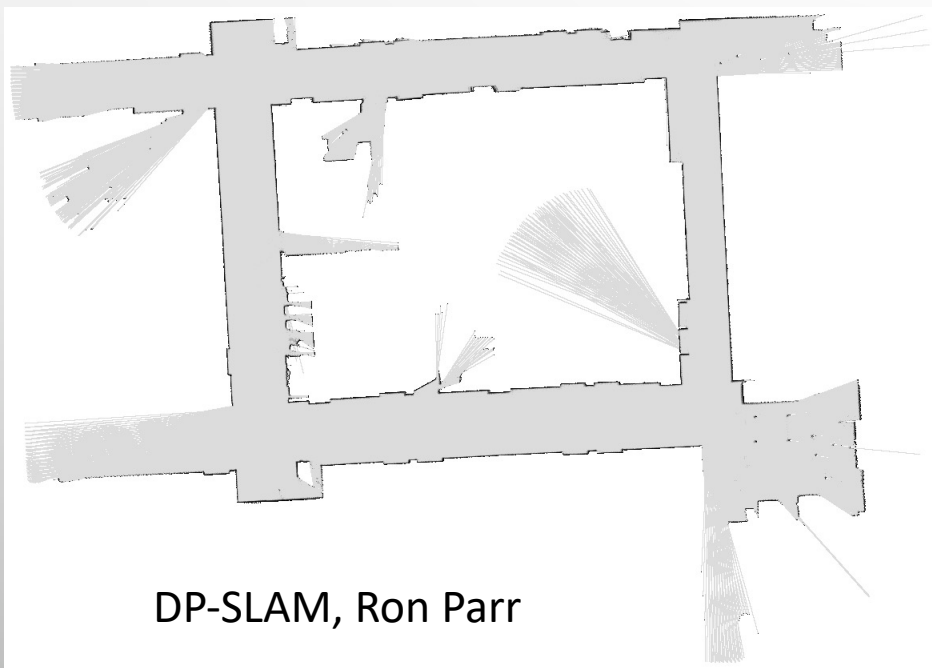


# Particle Filter Localization (Laser)

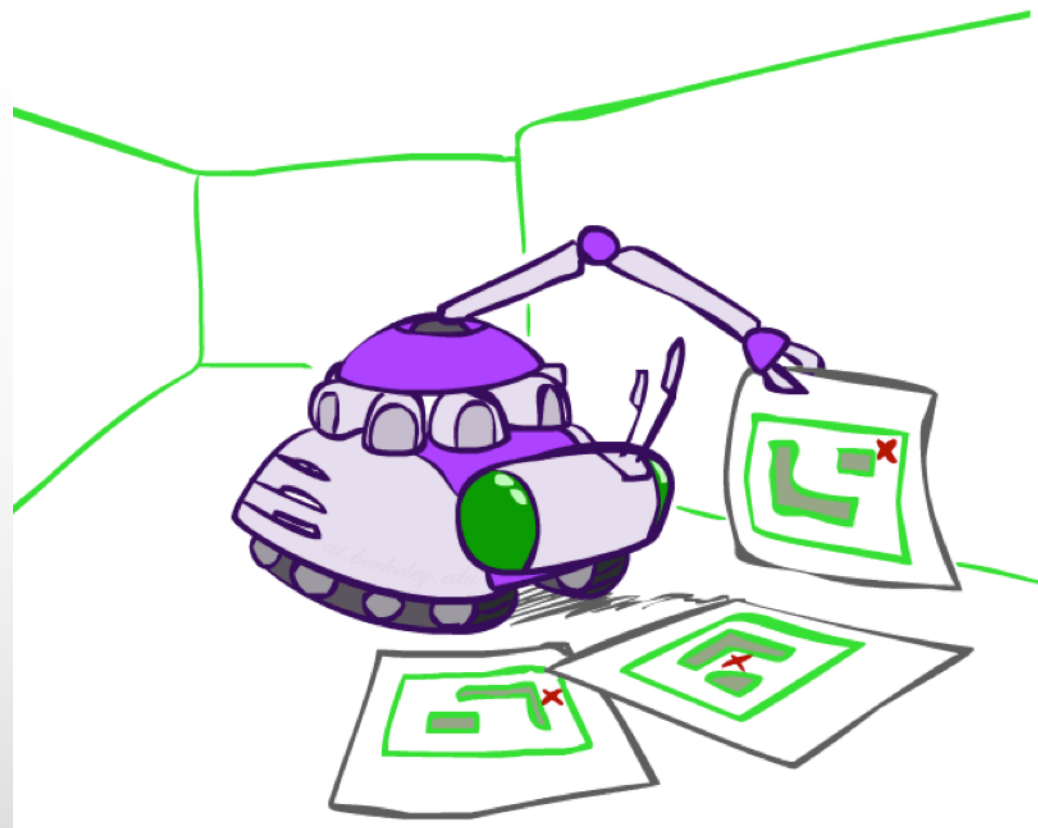


# Robot Mapping

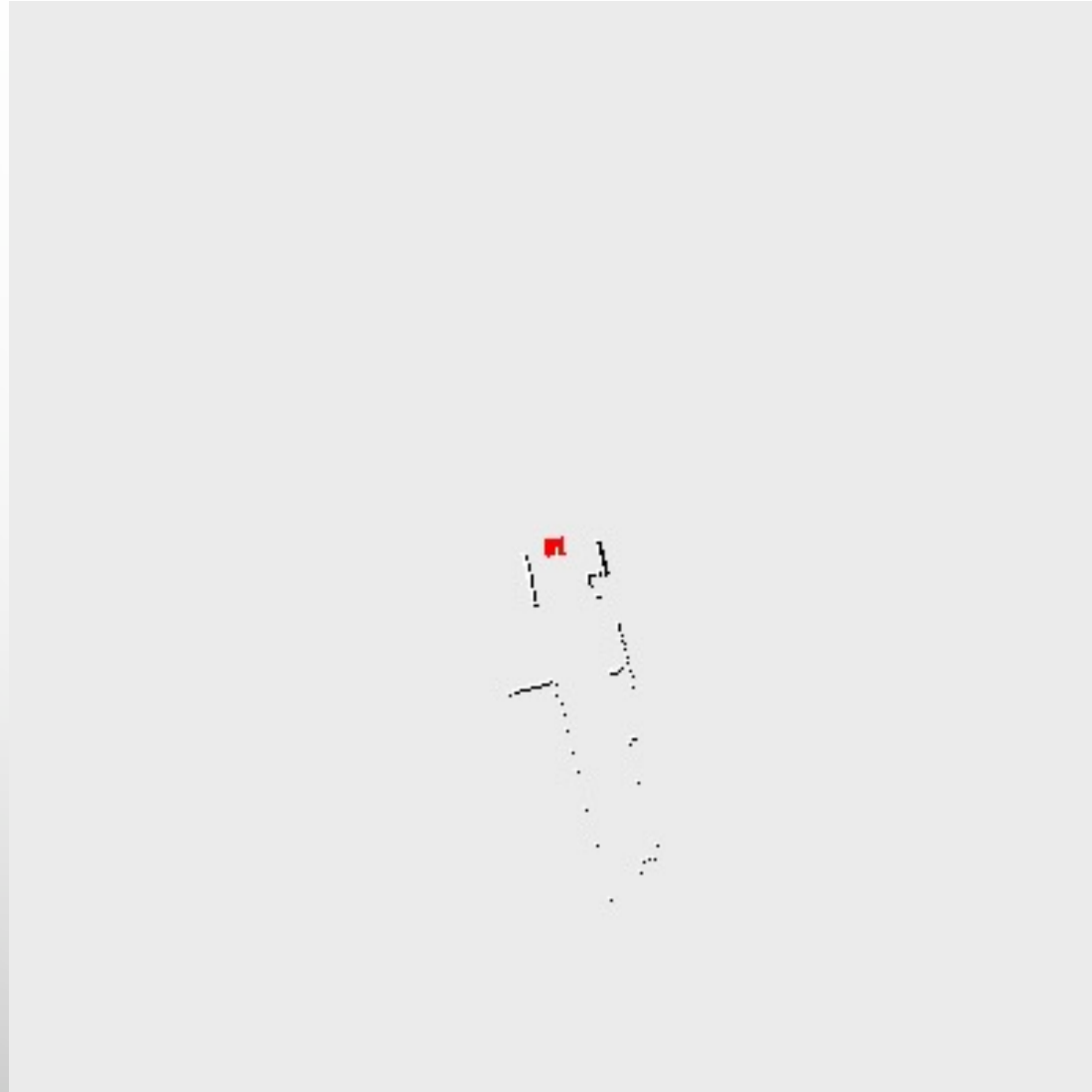
- SLAM: simultaneous localization and mapping
  - We do not know the map or our location
  - State consists of position AND map!
  - Main techniques: Kalman filtering (Gaussian HMMs) and particle methods



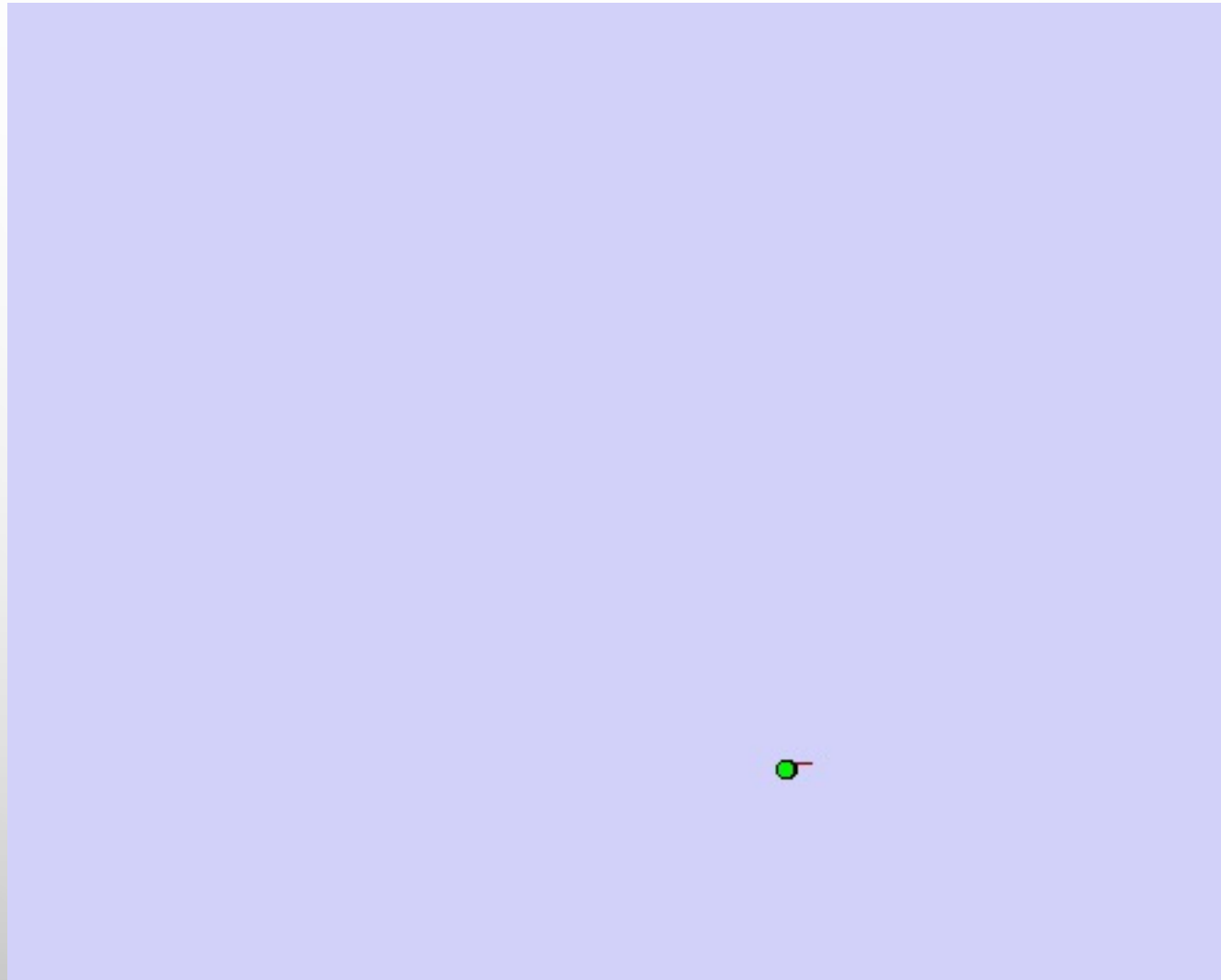
DP-SLAM, Ron Parr



# Particle Filter SLAM – Video 1

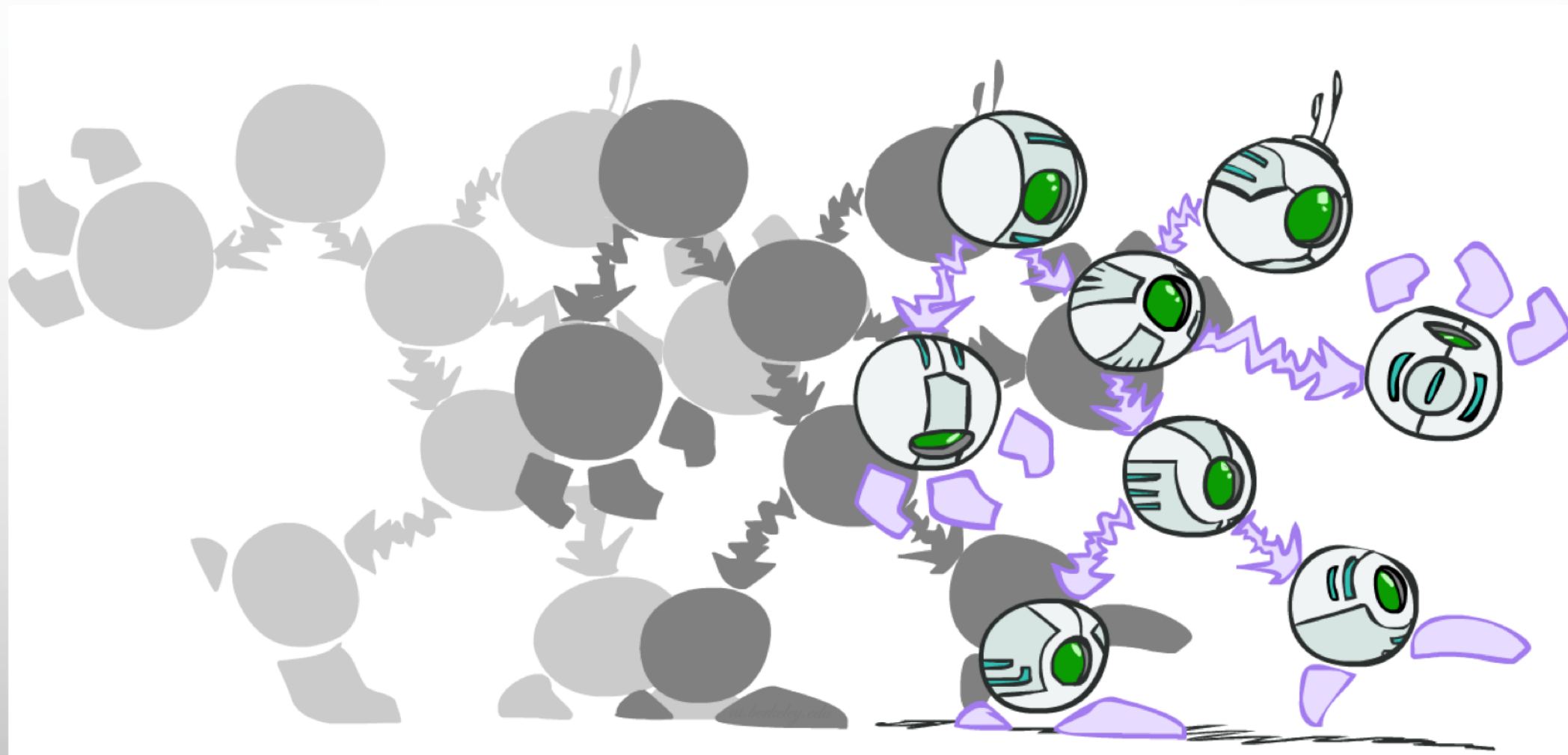


# Particle Filter SLAM – Video 2



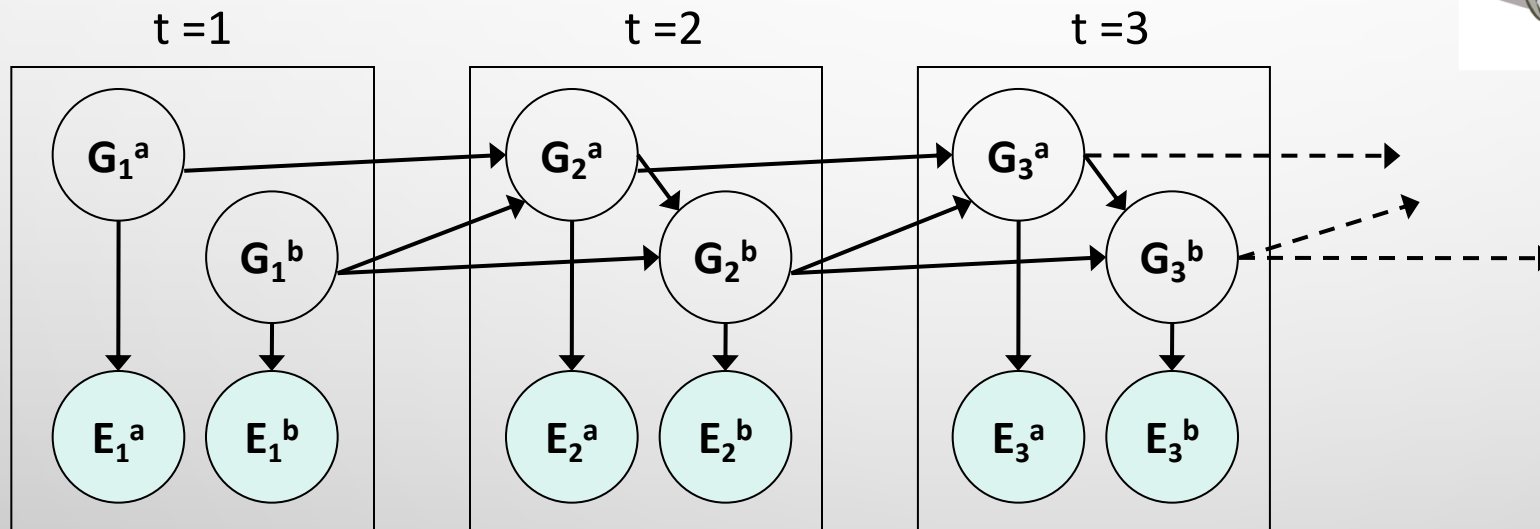
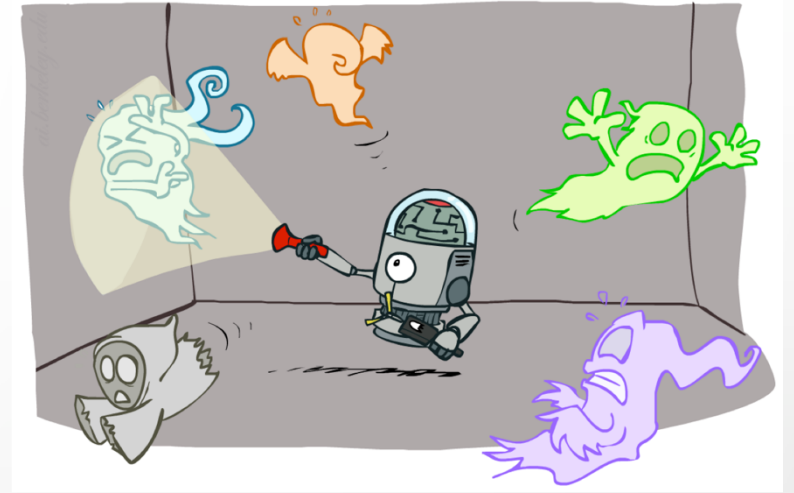


# Dynamic Bayes Nets



# Dynamic Bayes Nets (DBNs)

- We want to track multiple variables over time, using multiple sources of evidence
- Idea: repeat a fixed Bayes net structure at each time
- Variables from time  $t$  can condition on those from  $t-1$



- Dynamic Bayes nets are a generalization of HMMs

# DBN Particle Filters

- A particle is a complete sample for a time step
- **Initialize:** generate prior samples for the  $t=1$  Bayes net
  - Example particle:  $\mathbf{g}_1^a = (3,3)$   $\mathbf{g}_1^b = (5,3)$
- **EIapse time:** sample a successor for each particle
  - Example successor:  $\mathbf{g}_2^a = (2,3)$   $\mathbf{g}_2^b = (6,3)$
- **Observe:** weight each entire sample by the likelihood of the evidence conditioned on the sample
  - Likelihood:  $p(\mathbf{e}_1^a | \mathbf{g}_1^a) * p(\mathbf{e}_1^b | \mathbf{g}_1^b)$
- **Resample:** select prior samples (tuples of values) in proportion to their likelihood



# Most Likely Explanation



# HMMs: MLE Queries

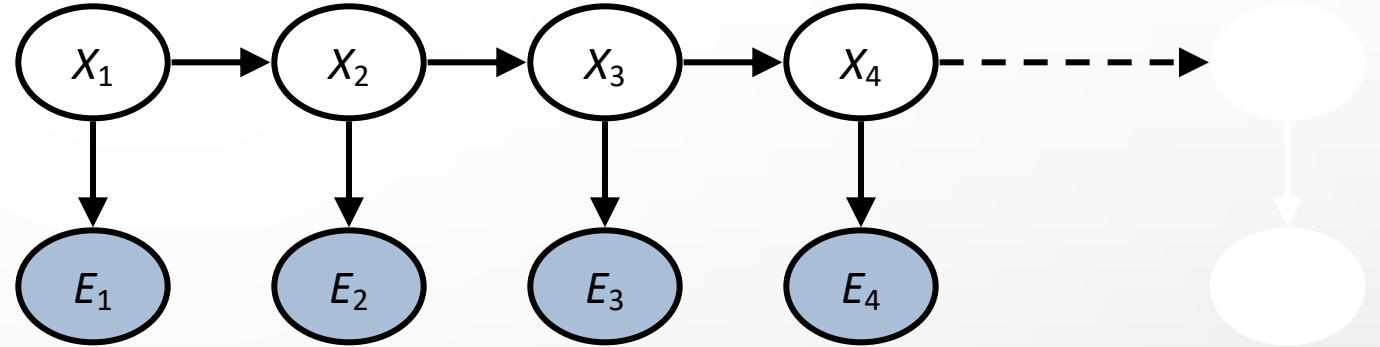
- HMMs defined by

- States  $X$
- Observations  $E$

- Initial distribution:  $P(X_1)$

- Transitions:  $P(X|X_{-1})$

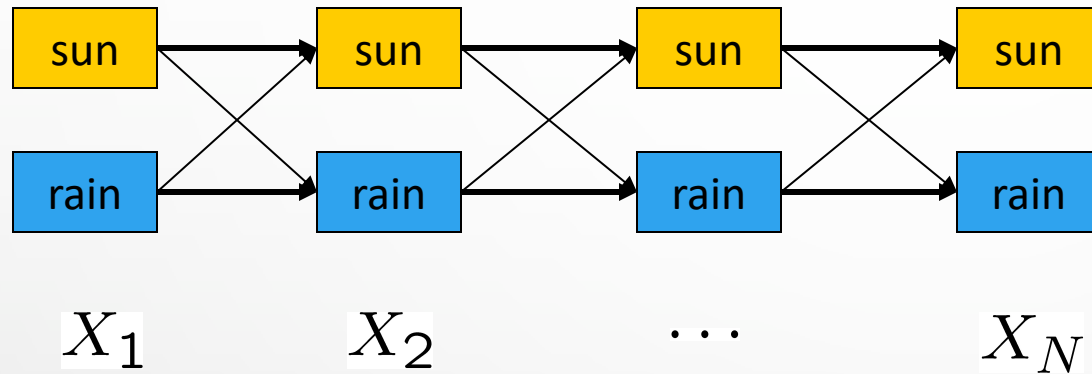
- Emissions:  $P(E|X)$



- New query: most likely explanation:  $\arg \max_{x_{1:t}} P(x_{1:t}|e_{1:t})$
- New method: the Viterbi algorithm

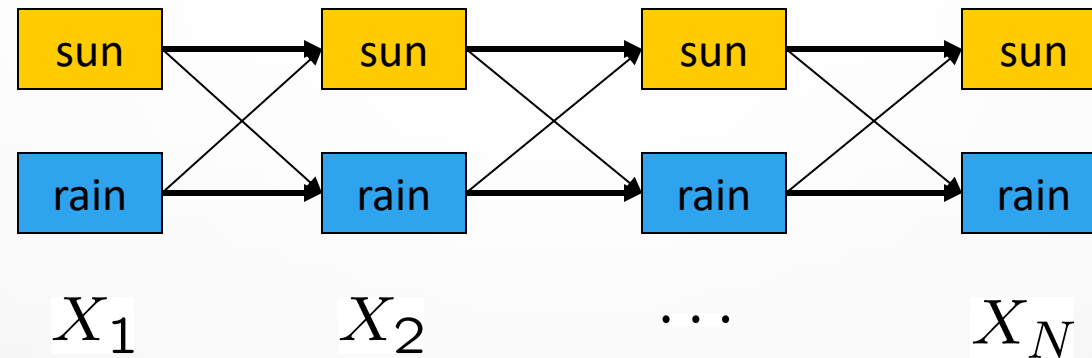
# State Trellis

- State trellis: graph of states and transitions over time



- Each arc represents some transition  $x_{t-1} \rightarrow x_t$
- Each arc has weight  $P(x_t|x_{t-1})P(e_t|x_t)$
- Each path is a sequence of states
- The product of weights on a path is that sequence's probability along with the evidence
- Forward algorithm computes sums of paths, Viterbi computes best paths

# Forward / Viterbi Algorithms



Forward Algorithm (Sum)

$$f_t[x_t] = P(x_t, e_{1:t})$$

$$= P(e_t|x_t) \sum_{x_{t-1}} P(x_t|x_{t-1}) f_{t-1}[x_{t-1}]$$

Viterbi Algorithm (Max)

$$m_t[x_t] = \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t})$$

$$= P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1}) m_{t-1}[x_{t-1}]$$



# Challenge

- Setting

- User we want to spy on use HTTPS to browse the internet

- Measurements

- IP address
- Sizes of packets coming in

- Goal

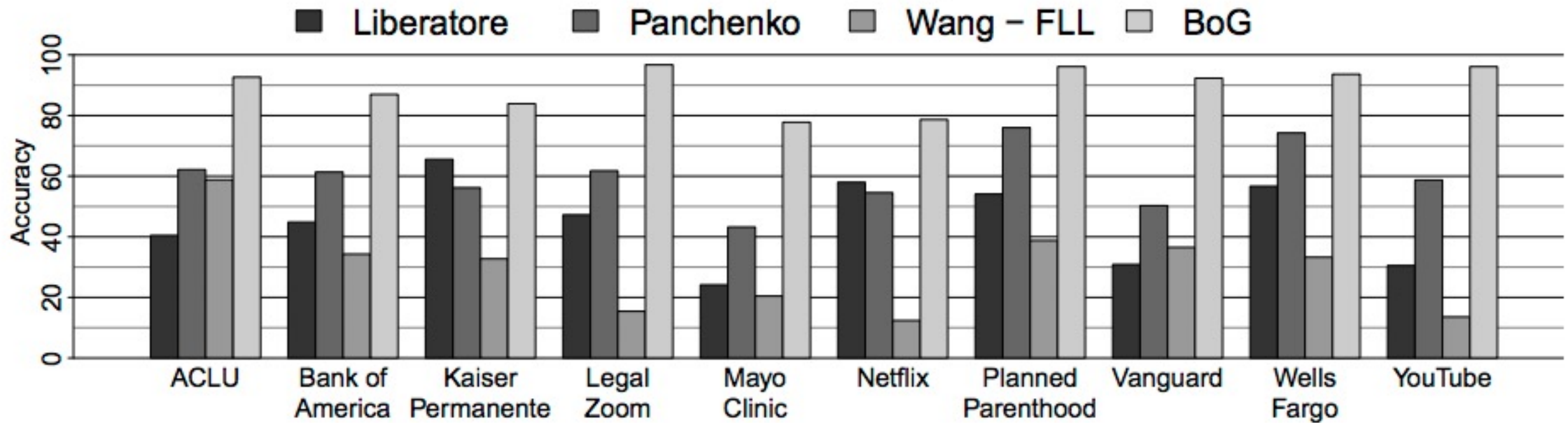
- Infer browsing sequence of that user

- e.g.: Medical, financial, legal, ...

# HMM

- Transition model
    - Probability distribution over links on the current page + some probability to navigate to any other page on the site
  - Noisy observation model due to traffic variations
    - Caching
    - Dynamically generated content
    - User-specific content, including cookies
- Probability distribution  $P(\text{packet size} \mid \text{page})$

# Results



BoG = described approach, others are prior work



# Results

## Session Length Effect

